

A Lam–Postnikov–Pylyavskyy inequality for hybrid Grothendieck polynomials

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Abstract

We prove a multivariate Lam–Postnikov–Pylyavskyy type inequality for hybrid Grothendieck polynomials, unifying and refining results for stable and dual stable Grothendieck polynomials established by Chan–Chen–Pak–Soskin. We also conjecture extensions of the Lam–Postnikov–Pylyavskyy inequality and a conjecture by Thomas–Yong to the (equivariant) Schubert and Grothendieck polynomial setting.

Keywords: Lam–Postnikov–Pylyavskyy inequality, Schur positivity, Schur polynomial, hybrid Grothendieck polynomial, multivariate Reuter–Lovász–Saks inequality, Schur orchestra inequality

AMS Classifications: 05E05, 05A20, 05E14, 14N15

1 Introduction

1.1 Lam–Postnikov–Pylyavskyy inequality

The Lam–Postnikov–Pylyavskyy (LPP) inequality [16] is a celebrated result in the Schur positivity and log-concavity phenomenon. For partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$, define their join $\lambda \vee \mu$ and meet $\lambda \wedge \mu$ as

$$(\lambda \vee \mu)_i = \max\{\lambda_i, \mu_i\}, \quad (\lambda \wedge \mu)_i = \min\{\lambda_i, \mu_i\}.$$

Let $s_\lambda(\mathbf{x})$ be the Schur function associated with λ . For two symmetric functions f and g , write $f \leq_s g$ if $g - f$ is Schur positive, namely,

$$g - f \in \sum_{\lambda} \mathbb{R}_{\geq 0} \cdot s_\lambda(\mathbf{x}).$$

The Schur LPP inequality states that

Theorem 1.1 ([16, Theorem 5]). *For partitions λ and μ , we have*

$$s_\lambda(\mathbf{x})s_\mu(\mathbf{x}) \leq_s s_{\lambda \vee \mu}(\mathbf{x})s_{\lambda \wedge \mu}(\mathbf{x}).$$

For more thorough information about LPP type inequalities as well as general Schur positivity and log-concavity, we refer to the recent breakthrough work by Speyer [24] and Chan–Chen–Pak–Soskin [4].

As an advanced extension of the LPP inequality, Chan, Chen, Pak and Soskin [4, Theorem 1.6] developed the Schur positivity version of the Ahlswede–Daykin inequality. As the main application, they resolved a Schur positivity conjecture concerning stable Grothendieck polynomials made by Mihalcea. Stable Grothendieck polynomials $G_\lambda(\mathbf{x})$ are central combinatorial objects in the study of Schubert calculus in the K-theory of Grassmannians. A notable combinatorial construction of $G_\lambda(\mathbf{x})$ was given by Buch [3] by using set-valued tableaux.

Let $\tilde{G}_\lambda(\mathbf{x}) = (-1)^{|\lambda|} G_\lambda(-\mathbf{x})$ be the modification such that each monomial has nonnegative coefficient. The following was conjectured by Mihalcea [21].

Theorem 1.2 ([4, Theorem 1.8]). *For partitions λ and μ , we have*

$$\tilde{G}_\lambda(\mathbf{x})\tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x})\tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

Recall that the Hall inner product on symmetric functions is defined by letting $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$. The orthogonal dual to $G_\lambda(\mathbf{x})$, denoted $g_\lambda(\mathbf{x})$, was systematically investigated by Lam and Pylyavskyy [17] which can be combinatorially generated by reverse plane partitions. The following dual stable Grothendieck LPP inequality is another application of the Ahlswede–Daykin–Schur inequality.

Theorem 1.3 ([4, Theorem 1.10]). *For partitions λ and μ , we have*

$$g_\lambda(\mathbf{x})g_\mu(\mathbf{x}) \leq_s g_{\lambda \vee \mu}(\mathbf{x})g_{\lambda \wedge \mu}(\mathbf{x}).$$

1.2 Main result

The hybrid Grothendieck polynomial $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$ with two families of parameters $\mathbf{t}$ and $\mathbf{w}$ is the generating function

$$H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w}) = \sum_{T \in \text{SVRPP}(\lambda)} \mathbf{x}^{\text{ircont}(T)} \mathbf{t}^{\text{ceq}(T)} \mathbf{w}^{\text{ex}(T)}$$

over set-valued reverse plane partitions of shape λ . This class of symmetric functions was introduced by the authors [13] originally in order to unify stable and dual stable Grothendieck polynomials. More specifically,

Remark 1.4. *We have the following specializations.*

- $t_i = 0$ for $i \geq 1$: $H_\lambda(\mathbf{x}; \mathbf{0}, \mathbf{w})$ is the refined stable Grothendieck polynomial $G_\lambda(\mathbf{x}; \mathbf{w})$ defined by Chan and Pflueger [6]. If further setting every $w_i = -1$, then $H_\lambda(\mathbf{x}; \mathbf{0}, -\mathbf{1})$ is equal to $G_\lambda(\mathbf{x})$. Note also that $H_\lambda(\mathbf{x}; \mathbf{0}, \mathbf{1}) = \tilde{G}_\lambda(\mathbf{x})$.
- $w_i = 0$ for $i \geq 1$: $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{0})$ is the refined dual stable Grothendieck polynomial $g_\lambda(\mathbf{x}; \mathbf{t})$ defined by Galashin, Grinberg and Liu [11]. If further setting every $t_i = 1$, then $H_\lambda(\mathbf{x}; \mathbf{1}, \mathbf{0})$ is equal to $g_\lambda(\mathbf{x})$.

Though known properties suggest that $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$ possesses remarkable properties, resembling those owned by stable and dual stable Grothendieck polynomial, some problems and conjectures are still open in the hybrid setting [13].

We extend the ordinary Schur positivity to a polynomial level. Suppose that f and g are symmetric functions in $\mathbf{x}$, with coefficients being polynomials in $\mathbf{z} = (z_1, z_2, \dots)$. Write $f \leq_s^{\mathbf{z}} g$ if

$$g - f \in \sum_{\lambda} \mathbb{R}_{\geq 0}[\mathbf{z}] \cdot s_\lambda(\mathbf{x}).$$

Our main result is stated as follows.

Theorem 1.5. *For partitions λ and μ , we have*

$$H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})H_\mu(\mathbf{x}; \mathbf{t}, \mathbf{w}) \leq_s^{\mathbf{t}, \mathbf{w}} H_{\lambda \vee \mu}(\mathbf{x}; \mathbf{t}, \mathbf{w})H_{\lambda \wedge \mu}(\mathbf{x}; \mathbf{t}, \mathbf{w}). \quad (1.1)$$

By Remark 1.4, we see that Theorem 1.5 unifies Theorems 1.2 and 1.3 for stable and dual stable Grothendieck polynomials.

Taking $\mathbf{t} = \mathbf{0}$ in $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$ gives the refined stable Grothendieck polynomial $G_\lambda(\mathbf{x}; \mathbf{w})$ of Chan and Pflueger [6]. In this case, Theorem 1.5 leads to a refinement of Theorem 1.2.

Corollary 1.6 (refined version of Theorem 1.2). *For partitions λ and μ , we have*

$$G_\lambda(\mathbf{x}; \mathbf{w})G_\mu(\mathbf{x}; \mathbf{w}) \leq_s^{\mathbf{w}} G_{\lambda \vee \mu}(\mathbf{x}; \mathbf{w})G_{\lambda \wedge \mu}(\mathbf{x}; \mathbf{w}).$$

Accordingly, taking $\mathbf{w} = \mathbf{0}$ in Theorem 1.5 yields a LPP inequality for the refined dual stable Grothendieck polynomial $g_\lambda(\mathbf{x}; \mathbf{t})$ of Galashin, Grinberg and Liu [11].

Corollary 1.7 (refined version of Theorem 1.3). *For partitions λ and μ , we have*

$$g_\lambda(\mathbf{x}; \mathbf{t})g_\mu(\mathbf{x}; \mathbf{t}) \leq_s^{\mathbf{t}} g_{\lambda \vee \mu}(\mathbf{x}; \mathbf{t})g_{\lambda \wedge \mu}(\mathbf{x}; \mathbf{t}).$$

Note that both Corollaries 1.6 and 1.7 specialize to Theorem 1.1 by taking $\mathbf{w} = \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$, respectively.

Our proof of Theorem 1.5 seems highly nontrivial. The arguments for Theorems 1.2 and 1.3 in [4] do not seem to directly apply to Theorem 1.5. Specifically, we will encounter two difficulties which will be briefly explained as follows. Write

$$H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w}) = \sum_{\mu} K_{\lambda, \mu}(\mathbf{t}, \mathbf{w}) s_{\mu}(\mathbf{x}).$$

- The proofs of Theorems 1.2 and 1.3 technically rely on tableau formulas for the coefficients in the Schur expansion of stable or dual stable Grothendieck polynomials [4]. There exists a combinatorial formula for the coefficients $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ using the language of lattice words [13, Theorem 1.3]. Unfortunately, this formula for $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ does not allow us to use similar arguments as in the proofs of Theorems 1.2 and 1.3. The essential obstruction is that the lattice word model lacks the structure of a distributive lattice.
- Another key tool used to prove Theorems 1.2 and 1.3 is the remarkable Ahlswede–Daykin–Schur inequality developed in [4, Theorem 1.6]. Intuitively, to acquire a proof of Theorem 1.5, we need a ‘multivariate’ version of the Ahlswede–Daykin–Schur inequality. However, we do not find an appropriate multivariate extension that is applicable to Theorem 1.5.

In order to address the above issues, we require substantial machinery.

- We establish a new formula for the coefficients $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ in terms of oscillating sequences, see Corollary 3.5. The idea behind this is to incorporate the operations of Bandlow and Morse [2] where the RSK and jeu de taquin algorithms are used to interpret the coefficients in the Schur expansion of stable and dual stable Grothendieck polynomials. The advantage is that oscillating sequences naturally form a distributive lattice, see Corollary 3.10, so that we could be able to use the framework that have been developed by Chan, Chen, Pak and Soskin [4] to deal with correlation inequalities.
- We use a multivariate Reuter–Lovász–Saks type inequality on finite distributive lattices, see Theorem 4.1, as well as a variation of the Schur orchestra inequality by Chan, Chen, Pak and Soskin [4, Theorem 5.2], see Theorem 5.2.

Remark 1.8. *In private communications, Swee Hong Chan pointed out that Theorem 4.1 can be implied (not obviously) by [5, Claim 6.3], and Theorem 5.2 follows from [4, Theorem 7.14]. Since the statement in [4, Theorem 7.14] is not needed in [4], its proof is omitted in [4]. For clarity and self-containedness, we will include proofs of Theorems 4.1 and 5.2 in this paper.*

Remark 1.9. *The Okounkov inequality for Schur functions was conjectured by Okounkov [23], and proved by Lam, Postnikov and Pylyavskyy [16, Theorem 4]:*

$$s_\lambda(\mathbf{x})s_\mu(\mathbf{x}) \leq_s s_{\lceil(\lambda+\mu)/2\rceil}(\mathbf{x})s_{\lfloor(\lambda+\mu)/2\rfloor}(\mathbf{x}).$$

Combining Theorems 1.2 and 1.3 with Speyer's theorem [24] (see [4, Theorem 3.5]), Chan, Chen, Pak and Soskin [4, Corollaries 9.5 and 10.3] obtained the Okounkov inequalities for stable and dual stable Grothendieck polynomials. We can also prove a multivariate Okounkov inequality for hybrid Grothendieck polynomials. The proof needs additional techniques, and will be presented in future work.

1.3 Conjectured extensions of the LPP inequality

In the Schubert calculus of the flag variety, Schur polynomials arise canonically as a subclass of Schubert polynomials, see Remark 1.10. It is therefore natural to ask if there exists a Schubert positivity extension which includes the classical Schur LPP inequality as a special case. Let S_n denote the symmetric group of permutations of $\{1, 2, \dots, n\}$. Write $w = w(1) \cdots w(n) \in S_n$ in one-line notation. The Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$, introduced by Lascoux and Schützenberger [18], represent the Schubert classes in the cohomology ring of the flag variety.

To define $\{\mathfrak{S}_w(\mathbf{x}) : w \in S_n\}$, recall the divided difference operator ∂_i acting on a polynomial $f(\mathbf{x})$ by letting $\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$, where $s_i \cdot f$ is obtained from f by interchanging the variables x_i and x_{i+1} . For $w_0 = n \cdots 21$, set $\mathfrak{S}_{w_0}(\mathbf{x}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. If $w \neq w_0$, then there must exist an index $1 \leq i < n$ such that $w(i) < w(i+1)$. Define $\mathfrak{S}_w(\mathbf{x}) = \partial_i \mathfrak{S}_{ws_i}(\mathbf{x})$, where ws_i is obtained from w by swapping $w(i)$ and $w(i+1)$. This is independent of the choice of i since the operators ∂_i satisfy the Coxeter relations.

The Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$ where $w \in S_n$ are linearly independent, constituting a basis of the space $\text{span}_{\mathbb{R}}\{x_1^{a_1} \cdots x_n^{a_n} : 0 \leq a_i \leq n - i\}$. For two polynomials f and g , write $f \leq_{\mathfrak{S}} g$ if $g - f$ is Schubert positive:

$$g - f \in \sum_w \mathbb{R}_{\geq 0} \cdot \mathfrak{S}_w.$$

An expected Schubert LPP inequality would look like

$$\mathfrak{S}_u(\mathbf{x})\mathfrak{S}_v(\mathbf{x}) \leq_{\mathfrak{S}} \mathfrak{S}_{u \vee v}(\mathbf{x})\mathfrak{S}_{u \wedge v}(\mathbf{x}). \quad (1.2)$$

The first task is how to define the join and meet of two permutations. To do this, we use the Lehmer code (also called inversion code) of w :

$$c(w) = (c_1(w), \dots, c_n(w)), \quad c_i(w) = \#\{j : i < j \leq n, w(i) > w(j)\}.$$

It is easily seen that $0 \leq c_i(w) \leq n - i$, and conversely every such integer vector determines a permutation in S_n . For $u, v \in S_n$, define their join and meet to be the permutations determined by the following Lehmer codes:

$$c_i(u \vee v) = \max\{c_i(u), c_i(v)\}, \quad c_i(u \wedge v) = \min\{c_i(u), c_i(v)\}.$$

Obviously, the above are indeed valid Lehmer codes of length n , and so $u \vee v$ and $u \wedge v$ are permutations belonging to S_n .

Remark 1.10. Let w be a k -Grassmannian, that is, it has at most one descent at position k . It is easy to check that $c_1(w) \leq \dots \leq c_k(w)$ and $c_i(w) = 0$ for $i > k$. Denote by $\lambda(w) = (c_k(w), \dots, c_1(w))$ the associated partition. It is a classical fact that $\mathfrak{S}_w(\mathbf{x})$ is the Schur polynomial $s_{\lambda(w)}(x_1, \dots, x_k)$. Notice that when u and v are k -Grassmannians, both $u \vee v$ and $u \wedge v$ are also k -Grassmannians, and moreover $\lambda(u \vee v) = \lambda(u) \vee \lambda(v)$ and $\lambda(u \wedge v) = \lambda(u) \wedge \lambda(v)$. Therefore, in such a case, (1.2) will reduce to the LPP inequality in Theorem 1.1.

For general u and v , we could immediately find examples that violate (1.2): for instance $u = 132$ and $v = 213$. A reasonable route is to consider permutations whose Schubert polynomials are close to Schur polynomials. Arguably, vexillary permutations are such permutations: w is vexillary if it avoids the pattern 2143, that is, there do not exist $i < j < k < \ell$ such that $w(j) < w(i) < w(\ell) < w(k)$. This class of permutations plays an important role in the combinatorics and geometry of Schubert varieties and degeneracy loci, see for example Fulton [9], Eriksson–Linusson [7] and Knutson–Miller–Yong [14].

Wachs' theorem [27] identifies the Schubert polynomial of a vexillary permutation with a flagged Schur polynomial. Precisely, for $w \in S_n$, set

$$I_i(w) = \{j : i < j \leq n, w(i) > w(j)\}.$$

So we have $c_i(w) = \#I_i(w)$. For every i with $c_i(w) > 0$, define

$$\beta_i(w) = \min I_i(w) - 1.$$

Arrange the entries $\{c_i(w) > 0\}$ in weakly decreasing order to get a partition $\lambda(w) = (\lambda_1 \geq \dots \geq \lambda_k)$, and the values $\{\beta_i(w) : c_i(w) > 0\}$ in weakly increasing order to get a flag $\phi(w) = (\phi_1 \leq \dots \leq \phi_k)$. When w is vexillary, it is well known that $\mathfrak{S}_w = s_{\lambda(w)}^{\phi(w)}$, the flagged Schur polynomial of shape $\lambda(w)$ and flagged by $\phi(w)$ [27].

Remark 1.11. A Schubert polynomial $\mathfrak{S}_w$ is a flagged Schur polynomial if and only if w is vexillary [18]. However, not every flagged Schur polynomial can be realized as a Schubert polynomial.

For two finite sequences $a = (a_1, \dots, a_r)$ and $b = (b_1, \dots, b_s)$, write $a \preceq b$ if a is a prefix of b , that is, $r \leq s$ and $a_i = b_i$ for $1 \leq i \leq r$.

Conjecture 1.12 (vexillary Schubert LPP inequality). Let $u, v \in S_n$ be vexillary permutations. Suppose that their Wachs flags satisfy $\phi(u) \preceq \phi(v)$. Then

$$\mathfrak{S}_u(\mathbf{x})\mathfrak{S}_v(\mathbf{x}) \leq_{\mathfrak{S}} \mathfrak{S}_{u \vee v}(\mathbf{x})\mathfrak{S}_{u \wedge v}(\mathbf{x}).$$

The above conjecture has been verified for $n \leq 8$.

Remark 1.13. Notice that Grassmannian permutations are vexillary permutations. The Wachs flag of a k -Grassmannian has all entries equal to k . Hence when both u and v are k -Grassmannians, Conjecture 1.12 would specialize to Theorem 1.1.

We give an example to illustrate Conjecture 1.12. Take two vexillary permutations $u = 5712364$ and $v = 3514672$ in S_7 , which have Lehmer codes

$$c(u) = (4, 5, 0, 0, 0, 1, 0), \quad c(v) = (2, 3, 0, 1, 1, 1, 0),$$

and Wachs flags

$$\phi(u) = (2, 2, 6) \preceq \phi(v) = (2, 2, 6, 6, 6).$$

So we have $u \vee v = 5713462$ and $u \wedge v = 3512476$. Note that $u \wedge v$ has the pattern 2143, and thus is no longer vexillary. By direct computation,

$$\mathfrak{S}_{u \vee v} \mathfrak{S}_{u \wedge v} - \mathfrak{S}_u \mathfrak{S}_v = 2 \mathfrak{S}_{(7,8,0,1,1,1)} + \mathfrak{S}_{(6,9,0,1,1,1)},$$

where, on the right-hand side, we use the Lehmer codes to indicate permutations (the former is a permutation in S_{10} , and the latter is in S_{11}).

We also formulate an equivariant version of Conjecture 1.12. The double Schubert polynomials $\mathfrak{S}_w(\mathbf{x}; \mathbf{y})$ represent the Schubert classes in the torus-equivariant cohomology of the flag variety. In the equivariant case, we set

$$\mathfrak{S}_{w_0}(\mathbf{x}; \mathbf{y}) = \prod_{i+j \leq n} (x_i - y_j).$$

If $w \neq w_0$, let $\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) = \partial_i \mathfrak{S}_{ws_i}(\mathbf{x}; \mathbf{y})$ where $w(i) < w(i+1)$. Here ∂_i acts on the $\mathbf{x}$ variables. It is clear that $\mathfrak{S}_w(\mathbf{x}) = \mathfrak{S}_w(\mathbf{x}; \mathbf{y})|_{y_i=0}$.

For $i \geq 1$, write $\alpha_i = y_{i+1} - y_i$. By Graham's positivity theorem [12], the structure constants $c_{u,v}^w(\mathbf{y})$ in the expansion of $\mathfrak{S}_u(\mathbf{x}; \mathbf{y})\mathfrak{S}_v(\mathbf{x}; \mathbf{y})$ satisfy

$$\mathfrak{S}_u(\mathbf{x}; \mathbf{y})\mathfrak{S}_v(\mathbf{x}; \mathbf{y}) = \sum_w c_{u,v}^w(\mathbf{y})\mathfrak{S}_w(\mathbf{x}; \mathbf{y}), \quad \text{where } c_{u,v}^w(\mathbf{y}) \in \mathbb{Z}_{\geq 0}[\alpha_1, \alpha_2, \dots].$$

This motivates the equivariant Schubert positivity: write $f \leq_{\mathfrak{S}}^{\text{equiv}} g$ if

$$g - f \in \sum_w \mathbb{R}_{\geq 0}[\alpha_1, \alpha_2, \dots] \cdot \mathfrak{S}_w(\mathbf{x}; \mathbf{y}).$$

Conjecture 1.14 (vexillary equivariant Schubert LPP inequality). *Let $u, v \in S_n$ be vexillary permutations. Suppose that their Wachs flags satisfy $\phi(u) \preceq \phi(v)$. Then*

$$\mathfrak{S}_u(\mathbf{x}; \mathbf{y})\mathfrak{S}_v(\mathbf{x}; \mathbf{y}) \leq_{\mathfrak{S}}^{\text{equiv}} \mathfrak{S}_{u \vee v}(\mathbf{x}; \mathbf{y})\mathfrak{S}_{u \wedge v}(\mathbf{x}; \mathbf{y}).$$

This conjecture will specialize to Conjecture 1.12 by taking $\mathbf{y} = \mathbf{0}$.

Because of the computational complexity of double Schubert polynomials, we tested Conjecture 1.14 randomly for vexillary permutations in S_n for $n = 9$.

When restricting w to a k -Grassmannian, $\mathfrak{S}_w(\mathbf{x}; \mathbf{y})$ is equal to the double Schur polynomial $s_{\lambda(w)}(\mathbf{x}_k; \mathbf{y})$, where $\mathbf{x}_k = (x_1, \dots, x_k)$, which is also called the factorial Schur polynomial, see for example [15, 22]. In the Grassmannian case, Conjecture 1.14 leads to an equivariant extension of Theorem 1.1.

Conjecture 1.15 (equivariant Schur LPP inequality). *For partitions λ and μ , we have*

$$s_{\lambda}(\mathbf{x}_k; \mathbf{y})s_{\mu}(\mathbf{x}_k; \mathbf{y}) \leq_s^{\text{equiv}} s_{\lambda \vee \mu}(\mathbf{x}_k; \mathbf{y})s_{\lambda \wedge \mu}(\mathbf{x}_k; \mathbf{y}).$$

This conjecture will specialize to the classical LPP inequality by taking $\mathbf{y} = \mathbf{0}$.

Conjecture 1.15 has been checked for all Grassmannians in S_n with n up to 8, and randomly for Grassmannians in S_{10} . It would be interesting to see whether (equivariant) geometric method could be brought to bear on this conjecture.

We lastly turn to the Grothendieck setting. As the K-theory counterpart, Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x})$ are polynomial representatives of Schubert classes in the Grothendieck ring of the flag variety [19]. We shall use the β -version $\mathfrak{G}_w^{(\beta)}(\mathbf{x})$ defined by Fomin and Kirillov [8], which specializes to $\mathfrak{S}_w(\mathbf{x})$ and $\mathfrak{G}_w(\mathbf{x})$ by taking $\beta = 0$ and $\beta = -1$, respectively.

The β -Grothendieck polynomials are defined in the same way as Schubert polynomials, except that we replace ∂_i by the operator $\pi_i^{(\beta)}(f) = \partial_i((1 + \beta x_{i+1})f)$. Now, for two polynomials f and g , write $f \leq_{\mathfrak{G}^{(\beta)}} g$ if

$$g - f \in \sum_w \mathbb{R}_{\geq 0}[\beta] \cdot \mathfrak{G}_w^{(\beta)}.$$

The following K-theory analogue of Conjecture 1.12 has been checked for $n \leq 7$.

Conjecture 1.16 (vexillary Grothendieck LPP inequality). *Let $u, v \in S_n$ be vexillary permutations. Suppose that their Wachs flags satisfy $\phi(u) \preceq \phi(v)$. Then*

$$\mathfrak{G}_u^{(\beta)}(\mathbf{x})\mathfrak{G}_v^{(\beta)}(\mathbf{x}) \leq_{\mathfrak{G}^{(\beta)}} \mathfrak{G}_{u \vee v}^{(\beta)}(\mathbf{x})\mathfrak{G}_{u \wedge v}^{(\beta)}(\mathbf{x}).$$

This conjecture will specialize to Conjecture 1.12 by taking $\beta = 0$.

Similarly to the Schubert case, when w is a vexillary permutation, $\mathfrak{G}_w^{(\beta)}(\mathbf{x})$ can be constructed by flagged set-valued tableaux, see for example [14]. In particular, for w a k -Grassmannian, its Grothendieck polynomial (for $\beta = 1$) is equal to $\tilde{G}_{\lambda(w)}(\mathbf{x}_k)$. Hence when restricted to k -Grassmannians, Conjecture 1.16 reduces to a statement for stable Grothendieck polynomials, as previously predicted by Thomas and Yong [26, Conjecture 9.2] (see also Mihalcea [21]) which remains open.

Conjecture 1.17 ([26, Conjecture 9.2]). *For partitions λ and μ , we have*

$$\tilde{G}_\lambda(\mathbf{x})\tilde{G}_\mu(\mathbf{x}) \leq_{\tilde{G}} \tilde{G}_{\lambda \vee \mu}(\mathbf{x})\tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

The double Grothendieck polynomials are the K-theory analogue of double Schubert polynomials. The structure constants in this setting are positive in the sense of Anderson, Griffeth and Miller [1]. The numerical data indicate that Conjectures 1.16 and 1.17 might also possess an equivariant Grothendieck LPP positivity.

The structure of this paper is as follows. In Section 2, we lay out necessary definitions and notation for this paper. In Section 3, we introduce the notion of oscillating sequences which is the starting point of our proof of Theorem 1.5. In particular, oscillating sequences form a distributive lattice. Sections 4 and 5 are devoted to a multivariate Reuter–Lovász–Saks type inequality and the Schur orchestra inequality, respectively. Collecting the results in Sections 3, 4 and 5, we complete the proof of Theorem 1.5 in Section 6.

Acknowledgements

We thank Swee Hong Chan for providing very valuable comments on a draft of this paper. This work was supported by the National Natural Science Foundation of China (No. 12371329) and the Fundamental Research Funds for the Central Universities (No. 63263094).

2 Preliminaries

In this section, we shall collect some terminology and notation. Some basic properties that we require will also be mentioned. In particular, we refer to [10, §1.1] and [25, Chapter 7] for the theory of symmetric functions.

2.1 Partition and Young lattice

A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers with only finitely many nonzero parts. Denote by $|\lambda| = \sum_i \lambda_i$ the size of λ , and by $\ell(\lambda)$ the number of nonzero parts of λ . The Young diagram of λ is a left-justified array of boxes with λ_i boxes in row i . Let (i, j) represent the box in row i and column j . Here the row (resp., column) indices are counted from top to bottom (resp., from left to right). Let $\mathcal{P}$ represent the set of all partitions. We use $\mu \subseteq \lambda$ to mean that the Young diagram of μ is contained in that of λ , namely, $\lambda_i \geq \mu_i$ for

all $i \geq 1$. Under the inclusion $\subseteq$ of partitions, $\mathcal{P}$ forms a distributive poset, called the Young lattice. Specifically, for two partitions λ and μ , their *join* $\lambda \vee \mu$ and *meet* $\lambda \wedge \mu$ are partitions satisfying

$$(\lambda \vee \mu)_i = \max\{\lambda_i, \mu_i\}, \quad (\lambda \wedge \mu)_i = \min\{\lambda_i, \mu_i\}.$$

For $\mu \subseteq \lambda$, the skew diagram λ/μ is obtained from the diagram λ by removing the boxes in μ . We say λ/μ is a *horizontal* (resp., *vertical*) *strip* if there are no two boxes lying in the same column (resp., row). Here we allow a horizontal or vertical strip to be empty (namely, in the case $\lambda = \mu$).

2.2 Schur and flagged Schur functions

Let λ/μ be a skew shape. A *semistandard Young tableau* of shape λ/μ is a filling $T: \lambda/\mu \rightarrow \mathbb{Z}_{>0}$ of the boxes of λ/μ with positive integers, such that the entries are weakly increasing from left to right along each row and strictly increasing from top to bottom along each column. In other words, whenever the corresponding boxes belong to λ/μ , we have

$$T(i, j) \leq T(i, j+1), \quad T(i, j) < T(i+1, j).$$

Denote by $\text{SSYT}(\lambda/\mu)$ the set of all semistandard tableaux of shape λ/μ . For $T \in \text{SSYT}(\lambda/\mu)$, its weight is the sequence $\text{wt}(T) = (\text{wt}_1(T), \text{wt}_2(T), \dots)$, where $\text{wt}_i(T)$ is the number of entries of T equal to i . Write

$$\mathbf{x}^{\text{wt}(T)} = \prod_{i \geq 1} x_i^{\text{wt}_i(T)}.$$

The *Schur function* indexed by skew shape λ/μ is defined by

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \text{SSYT}(\lambda/\mu)} \mathbf{x}^{\text{wt}(T)}.$$

In the straight shape case when $\mu = \emptyset$, we write $s_\lambda(\mathbf{x})$ instead of $s_{\lambda/\emptyset}(\mathbf{x})$. If we use a finite number of variables $\mathbf{x}_n = (x_1, \dots, x_n)$, then we obtain a Schur polynomial. It is a classical result that the set of Schur functions forms a linear basis of the ring of symmetric functions.

The flagged Schur polynomials are generalizations of Schur polynomials. A *flag* means a weakly increasing sequence $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ of positive integers. Suppose that λ/μ has at most r rows. A *flagged semistandard tableau* of shape λ/μ and with flag ϕ is a semistandard tableau of shape λ/μ such that every entry in row i cannot exceed ϕ_i . Denote the set of such tableaux by $\text{SSYT}(\lambda/\mu, \phi)$. The corresponding *flagged Schur function* is

$$s_{\lambda/\mu}^\phi(\mathbf{x}) = \sum_{T \in \text{SSYT}(\lambda/\mu, \phi)} \mathbf{x}^{\text{wt}(T)}.$$

2.3 Stable and dual stable Grothendieck polynomials

Stable Grothendieck polynomials $G_\lambda(\mathbf{x})$ and their duals $g_\lambda(\mathbf{x})$ are nonhomogeneous generalizations of Schur functions. We first recall the set-valued tableau construction of $G_\lambda(\mathbf{x})$ pioneered by Buch [3]. For two nonempty finite sets $A, B \subset \mathbb{Z}_{>0}$, we write

$A \leq B$ (resp., $A < B$) to mean that $\max A \leq \min B$ (resp., $\max A < \min B$). A *set-valued tableau* of shape λ/μ is a filling T of each box of λ/μ with a nonempty finite subset of $\mathbb{Z}_{>0}$ such that the sets are weakly increasing along each row, and strictly increasing down each column. Let $\text{SVT}(\lambda/\mu)$ be the set of all set-valued tableaux of shape λ/μ .

For $T \in \text{SVT}(\lambda/\mu)$, define $|T| = \sum_{(i,j) \in \lambda/\mu} \#T(i,j)$. Its weight is $\text{wt}(T) = (\text{wt}_1(T), \text{wt}_2(T), \dots)$, where $\text{wt}_i(T)$ is the total number of occurrences of i appearing in T . The *stable Grothendieck polynomial* of skew shape λ/μ is defined by

$$G_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \text{SVT}(\lambda/\mu)} (-1)^{|T| - |\lambda/\mu|} \mathbf{x}^{\text{wt}(T)}.$$

It is easy to see that the Schur function s_λ is equal to the lowest-degree homogeneous component of $G_\lambda(\mathbf{x})$.

The Hall inner product on symmetric functions is defined by setting $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$. As the dual basis of $G_\lambda(\mathbf{x})$, the *dual stable Grothendieck polynomials* $g_\lambda(\mathbf{x})$ were extensively studied by Lam and Pylyavskyy [17]. Unlike $G_\lambda(\mathbf{x})$, the dual $g_\lambda(\mathbf{x})$ is combinatorially constructed in terms of reverse plane partitions. Recall that a *reverse plane partition* of shape λ/μ is a filling $T: \lambda/\mu \rightarrow \mathbb{Z}_{>0}$ of the boxes in λ/μ such that the entries are weakly increasing along each row and each column. Denote by $\text{RPP}(\lambda/\mu)$ the set of reverse plane partitions of shape λ/μ .

For $T \in \text{RPP}(\lambda/\mu)$, the irredundant content vector $\text{ircont}(T) = (r_1, r_2, \dots)$ is defined by letting r_i be the number of columns that contain at least one i . Define

$$g_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \text{RPP}(\lambda/\mu)} \mathbf{x}^{\text{ircont}(T)}.$$

It follows from the definition that the highest-degree homogeneous component of $g_\lambda(\mathbf{x})$ is the Schur function s_λ .

Lam and Pylyavskyy [17] showed that $\langle G_\lambda, g_\mu \rangle = \delta_{\lambda,\mu}$. Importantly, this relation yields an alternative way to compute the structure constants in the K-theory of Grassmannians, see also Li, Morse and Shields [20].

The expansions of G_λ and g_λ in the basis of Schur functions have been given in [3] and [17], respectively, see also Bandlow and Morse [2]. The coefficients appearing in the expansion have explicit combinatorial formulas in terms of certain kinds of tableaux. Such formulas are crucial in the proofs of Theorems 1.2 and 1.3, as given by Chan, Chen, Pak and Soskin [4].

2.4 Hybrid Grothendieck polynomials

The hybrid Grothendieck polynomials $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$, which are defined based on set-valued reverse plane partitions, provide a unified platform to investigate stable and dual stable Grothendieck polynomials. A *set-valued reverse plane partition* of shape λ/μ is a filling of every box of λ/μ with a nonempty finite set of positive integers, such that the sets are weakly increasing along each row and each column. See Figure 1 for an illustration. The set of such fillings is denoted by $\text{SVRPP}(\lambda/\mu)$.

For $T \in \text{SVRPP}(\lambda/\mu)$ and a box $(i, j) \in \lambda/\mu$, let $T(i, j)$ be the set in (i, j) . To define $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$, we need the following statistics:

- the *irredundant content vector* $\text{ircont}(T) = (r_1, r_2, \dots)$, where r_i is the number of columns containing a set which contains i ;

			2
	1	12	23
12	23	3	34
23	345	5	
35			

Figure 1: A set-valued reverse plane partition. Here we have omitted the braces $\{ \}$ for sets as well as the commas between numbers.

- the *column equality vector* $\text{ceq}(T) = (c_1, c_2, \dots)$, where c_i is the number of boxes (i, j) in row i such that $(i+1, j) \in \lambda/\mu$ and $\max T(i, j) = \min T(i+1, j)$;
- the *excess vector* $\text{ex}(T) = (e_1, e_2, \dots)$, where

$$e_i = \sum_{(i,j) \in \lambda/\mu} (\#T(i, j) - 1).$$

For the set-valued reverse plane partition in Figure 1, we have

$$\text{ircont}(T) = (3, 4, 4, 2, 3), \quad \text{ceq}(T) = (1, 1, 2, 1), \quad \text{ex}(T) = (0, 2, 3, 3, 1).$$

The *hybrid Grothendieck polynomial* of shape λ/μ is

$$H_{\lambda/\mu}(\mathbf{x}; \mathbf{t}, \mathbf{w}) = \sum_{T \in \text{SVRPP}(\lambda/\mu)} \mathbf{x}^{\text{ircont}(T)} \mathbf{t}^{\text{ceq}(T)} \mathbf{w}^{\text{ex}(T)}.$$

We emphasize that $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{t} = (t_1, t_2, \dots)$ and $\mathbf{w} = (w_1, w_2, \dots)$ are countable sequences of variables, and $\mathbf{y}^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \dots$ for a weak composition α .

As aforementioned in Remark 1.4, hybrid Grothendieck polynomials unify (refined) stable and dual stable Grothendieck polynomials. If one sets all $t_i = 0$, then $H_\lambda(\mathbf{x}; \mathbf{0}, \mathbf{w}) = G_\lambda(\mathbf{x}; \mathbf{w})$ is the *refined stable Grothendieck polynomial* due to Chan and Pflueger [6]. If one further sets all $w_i = -1$, then $H_\lambda(\mathbf{x}; \mathbf{0}, -\mathbf{1})$ is $G_\lambda(\mathbf{x})$. We also have $H_\lambda(\mathbf{x}; \mathbf{0}, \mathbf{1}) = \tilde{G}_\lambda(\mathbf{x})$. On the other hand, if one sets all $w_i = 0$, then $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{0}) = g_\lambda(\mathbf{x}; \mathbf{t})$ is the *refined dual stable Grothendieck polynomial* introduced by Galashin, Grinberg and Liu [11]. If one further sets all $t_i = 1$, then $H_\lambda(\mathbf{x}; \mathbf{1}, \mathbf{0})$ becomes $g_\lambda(\mathbf{x})$.

It was shown in [13, Theorem 1.2] that $H_{\lambda/\mu}(\mathbf{x}; \mathbf{t}, \mathbf{w})$ is indeed symmetric in $\mathbf{x}$. Moreover, $H_{\lambda/\mu}(\mathbf{x}; \mathbf{t}, \mathbf{w})$ is Schur positive. Using the crystal basis method, the coefficients $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ in the Schur expansion

$$H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w}) = \sum_{\mu} K_{\lambda, \mu}(\mathbf{t}, \mathbf{w}) s_{\mu}(\mathbf{x}) \tag{2.1}$$

can be formulated in terms of set-valued reverse plane partitions satisfying the lattice word condition [13, Theorem 1.3]. Note that [13, Theorem 1.3] is valid for general skew shapes. This paper only concerns the straight shape situation.

The formula of $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ given in [13, Theorem 1.3] is not applicable for our purpose of proving Theorem 1.5, because of the absence of a distributive lattice structure. To address this problem, inspired by the work of Bandlow and Morse [2], we design a dilation–contraction algorithm on set-valued reverse plane partitions to give a new formula for $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$, see Section 3.

2.5 RSK algorithm

The *RSK algorithm* inserts a positive integer x to a semistandard tableau T of straight shape, yielding a new semistandard tableau. The algorithm is described in a recursive procedure. First, insert x into the first row of T . If all entries in the first row are less than or equal to x , then place x in a new box at the end of first row and stop. Otherwise, let y be the smallest entry in the first row satisfying $y > x$. Replace y by x and insert the bumped entry y into the second row. Repeating the same procedure, the row insertion algorithm eventually terminates, giving rise to a new semistandard tableau, denoted $T \leftarrow x$. Note that the shape of $T \leftarrow x$ has one more extra corner, denoted B , than the shape of T . The RSK algorithm is reversible in the sense that one can uniquely recover T and x from $T \leftarrow x$ and the position of the extra corner box B . The collection of boxes where an entry is bumped, together with the corner box where the last bumped entry lands, is called the *bumping route*. Figure 2 gives an illustration of the RSK insertion for $x = 3$, where the boxes on the bumping route are shaded.

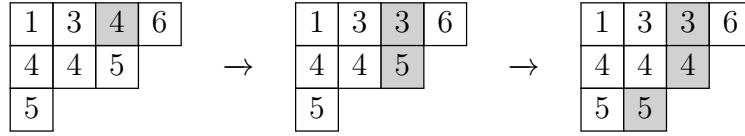


Figure 2: An illustration of the RSK algorithm.

The following property is not hard to justify, see [10, Chapter 1].

Proposition 2.1. *Let T be a semistandard tableau of shape λ , and let*

$$U = (((T \leftarrow x_1) \leftarrow x_2) \cdots) \leftarrow x_p$$

be obtained by successively inserting $x_1, \dots, x_p$. Let μ be the shape of U . If $x_1 > x_2 > \cdots > x_p$, then μ/λ forms a vertical strip. Conversely, suppose that U is a semistandard tableau of shape μ , and that $\lambda \subseteq \mu$ such that μ/λ has p boxes. If μ/λ is a vertical strip, then there exist a unique semistandard tableau T of shape λ and a unique sequence $x_1 > x_2 > \cdots > x_p$ such that

$$U = (((T \leftarrow x_1) \leftarrow x_2) \cdots) \leftarrow x_p.$$

2.6 Jeu de taquin algorithm

Let λ/μ be a skew diagram. An *inside corner* of λ/μ is a box which may be removed from μ while still leaving a valid Young diagram, while an *outside corner* of λ/μ is a box which may be removed from λ while still leaving a Young diagram. In Figure 3, the inside and outside corners are marked with $\circ$ and $\times$, respectively.

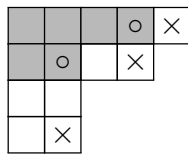


Figure 3: Inside and outside corners.

Let S be a semistandard tableau of shape λ/μ . Suppose that C is an inside corner of λ/μ . Regard C as an empty box. The *jeu de taquin (jdt) slide* starting from C is defined as follows. At each step, compare the entries in the two boxes immediately to the right and immediately below the empty box. Move the smaller of these two entries into the empty cell. If only one of the two neighbors exists, move that entry into the empty cell. If both neighbors exist and have equal entries, move the lower entry into the empty cell. The empty box has now moved to the position of the entry just moved. This process is repeated until the empty box reaches an outside corner, yielding a new semistandard tableau. The jdt slide is also reversible. Suppose that S' is the semistandard tableau obtained after a jdt slide, and that the outside corner B removed at the end of the slide is specified. Replace B by an empty box. One can reverse the slide procedure to recover the original tableau S . For example, in Figure 4, we apply the jdt algorithm to the inside corner $(2, 2)$ where the black boxes constitute the slide path.

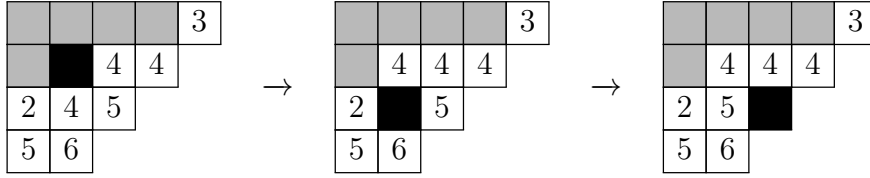


Figure 4: An illustration of the jdt slide.

3 Oscillating sequences

In this section, we introduce oscillating sequences to give a new combinatorial formula for the coefficients $K_{\lambda,\mu}(\mathbf{t}, \mathbf{w})$ appearing in the expansion in (2.1). The construction is based on the RSK and jeu de taquin algorithms. As a key property that we need, oscillating sequences admit a distributive lattice structure.

3.1 Oscillating sequences

Recall that a skew shape λ/μ is a horizontal (resp., vertical) strip if there are no two boxes lying in the same column (resp., row). Here we allow a horizontal or vertical strip to be empty. A pair of partitions (λ, μ) is called a *k-falling pair* if

- (1) $\lambda \supseteq \mu$ and $\lambda_i = \mu_i$ for all $i \leq k$,
- (2) λ/μ is a horizontal strip.

Dually, we call a pair of partitions (λ, μ) a *k-rising pair* if

- (1') $\lambda \subseteq \mu$ and $\lambda_i = \mu_i$ for all $i \leq k$,
- (2') μ/λ is a vertical strip.

Definition 3.1. Let λ and μ be partitions. For $l \geq 1$, an *l-oscillating sequence* from λ to μ is a sequence $S = (\lambda^0, \lambda^1, \dots, \lambda^{2l-1})$ of partitions such that

- (1) $\lambda^0 = \lambda$ and $\lambda^{2l-1} = \mu$;

- (2) For each $1 \leq i \leq l$, the pair $(\lambda^{2i-2}, \lambda^{2i-1})$ is an $(l-i+1)$ -rising pair, and for each $1 \leq j \leq l-1$, the pair $(\lambda^{2j-1}, \lambda^{2j})$ is an $(l-j)$ -falling pair.

Remark 3.2. In the remainder of this paper, we shall only be concerned with l -oscillating sequences from λ to μ where $l \geq \ell(\lambda)$.

For example, we take $\lambda = (4, 3, 2)$, $\mu = (4, 3, 3, 1)$, and $l = 3$. Figure 5 gives an 3-oscillating sequence from λ to μ .

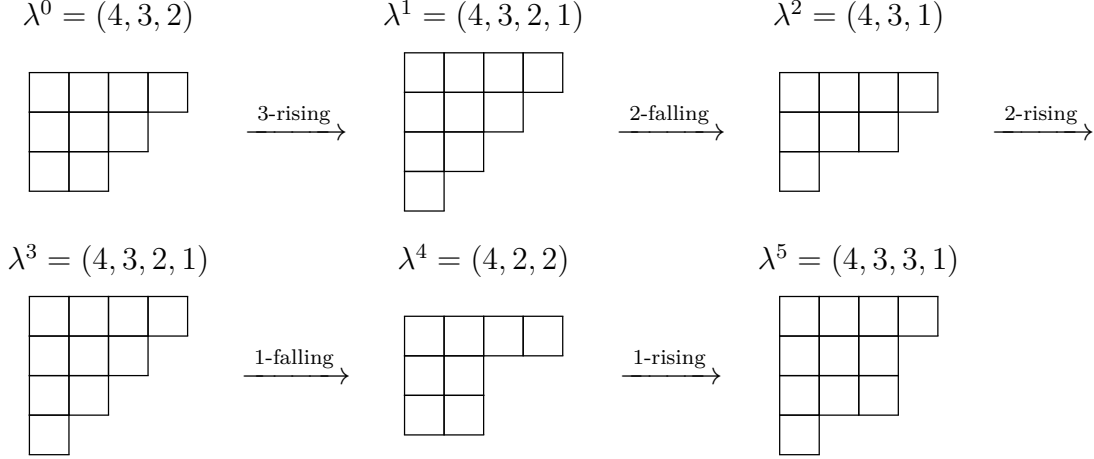


Figure 5: An oscillating sequence from $(4, 3, 2)$ to $(4, 3, 3, 1)$ with $l = 3$.

Remark 3.3. By definition, it is easy to see that for an oscillating sequence $S = (\lambda^0, \lambda^1, \dots, \lambda^{2l-1})$, the first parts of λ^i ($0 \leq i \leq 2l-1$) are of the same size.

Let $\text{OS}^l(\lambda \rightarrow \mu)$ denote the set of all l -oscillating sequences from λ to μ . Note that for fixed λ and μ , the set $\text{OS}^l(\lambda \rightarrow \mu)$ is finite since there are finitely many choices for the added vertical strips or the deleted horizontal strips in the construction process. For an oscillating sequence $S = (\lambda^0, \dots, \lambda^{2l-1})$, define

$$\text{wt}(S) = \prod_{1 \leq i \leq l} w_{l-i+1}^{a_i} \prod_{1 \leq j \leq l-1} t_{l-j}^{b_j}, \quad a_i = |\lambda^{2i-1}/\lambda^{2i-2}|, \quad b_j = |\lambda^{2j-1}/\lambda^{2j}|.$$

We relate set-valued reverse plane partitions to oscillating sequences in the following theorem.

Theorem 3.4. For any fixed partition λ , there exists a bijection

$$\Phi : \text{SVRPP}(\lambda) \longrightarrow \bigsqcup_{\mu} \text{SSYT}(\mu) \times \text{OS}^{\ell(\lambda)}(\lambda \rightarrow \mu).$$

Moreover, for $T \in \text{SVRPP}(\lambda)$, if $\Phi(T) = (T', S)$, then

$$\mathbf{x}^{\text{ircont}(T)} \mathbf{w}^{\text{ex}(T)} \mathbf{t}^{\text{ceq}(T)} = \mathbf{x}^{\text{wt}(T')} \text{wt}(S).$$

As a direct consequence of Theorem 3.4, we obtain a formula for the coefficients $K_{\lambda, \mu}(\mathbf{t}, \mathbf{w})$ in terms of oscillating sequences.

Corollary 3.5. For partitions λ and μ , we have

$$K_{\lambda, \mu}(\mathbf{t}, \mathbf{w}) = \sum_{\alpha} \sum_{\beta} f_{\lambda, \mu}^{(\alpha, \beta)} \mathbf{w}^{\alpha} \mathbf{t}^{\beta},$$

where

$$f_{\lambda, \mu}^{(\alpha, \beta)} = \#\{S \in \text{OS}^{\ell(\lambda)}(\lambda \rightarrow \mu) : \text{wt}(S) = \mathbf{w}^{\alpha} \mathbf{t}^{\beta}\}.$$

To prove Theorem 3.4, we introduce an algorithm, called the *dilation-contraction algorithm*, performed on set-valued reverse plane partitions. The idea behind this is mainly inspired by the operations separately on set-valued tableaux and reverse plane partitions investigated by Bandlow and Morse [2].

We first define the dilation operation. For a vector $v = (v_1, v_2, \dots)$ and $r \geq 1$, we use $v_{\geq r}$ to denote the truncation $v_{\geq r} = (v_r, v_{r+1}, \dots)$ starting from position r . For $r, s \geq 1$, let $\text{SVRPP}(\lambda, r, s)$ denote the subset of $\text{SVRPP}(\lambda)$ consisting of set-valued reverse plane partitions with $\text{ceq}_{\geq r}(T) = \mathbf{0}$ and $\text{ex}_{\geq s}(T) = \mathbf{0}$. In other words, every box of T in or below row r has no redundant entry, and every box of T in or below row s is filled with a single number.

For $k \geq 1$, the k -th dilation operation, denoted di_k , is performed on the set $\text{SVRPP}(\lambda, k, k+1)$. Let $T \in \text{SVRPP}(\lambda, k, k+1)$. We use $T_{\geq k+1}$ to denote the subtableau of T occupying the boxes in and below row $k+1$. Keep in mind that $T_{\geq k+1}$ itself is an ordinary semistandard tableau. The image $\text{di}_k(T)$ of k -th dilation acting on T is defined as follows.

- Consider the entries in the k -th row of T by ignoring the smallest entry in each box. Suppose that such entries are $x_1 > \dots > x_m$. Remove the entries $x_1, \dots, x_m$ from largest to smallest, and meanwhile insert these entries successively into $T_{\geq k+1}$ by applying the RSK algorithm.

For example, in Figure 6, we apply the dilation algorithm to a set-valued reverse plane partition of shape $\lambda = (2, 2, 1, 1)$ and with $k = 2$. In the process, we insert $(x_1, x_2, x_3) = (4, 3, 2)$ successively into $T_{\geq 3}$.

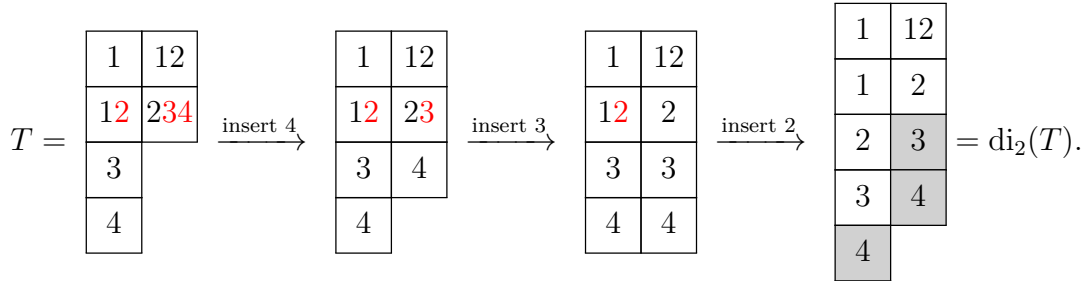


Figure 6: An illustration of the dilation algorithm.

Lemma 3.6. *Fix a shape λ . The map di_k is a bijection from $\text{SVRPP}(\lambda, k, k+1)$ to the union of $\text{SVRPP}(\mu, k, k)$, where (λ, μ) is a k -rising pair.*

Proof. Let $T \in \text{SVRPP}(\lambda, k, k+1)$, and $T' = \text{di}_k(T)$. We first check that T' belongs to $\text{SVRPP}(\mu, k, k)$, where (λ, μ) is a k -rising pair. Assume that $x_1 > \dots > x_m$ are the entries row k , as used in the definition of the dilation. Let P_1 be obtained from T by removing x_1 and then inserting x_1 into $T_{\geq k+1}$. It is straightforward to check that P_1 is a valid set-valued reverse plane partition with $\text{ceq}_{\geq k}(P_1) = \mathbf{0}$. Doing the same for $x_2, \dots, x_m$ yields the image T' which is also a valid set-valued reverse plane partition with $\text{ceq}_{\geq k}(T') = \mathbf{0}$. It is also clear that $\text{ex}_{\geq k}(T') = \mathbf{0}$. In view of Proposition 2.1, the difference of shapes of T' and T is a vertical strip. So we conclude that $T' \in \text{SVRPP}(\mu, k, k)$, where (λ, μ) is a k -rising pair.

Conversely, let $T' \in \text{SVRPP}(\mu, k, k)$, where (λ, μ) is a k -rising pair. Do the reverse RSK to the entries in the vertical strip μ/λ from bottom to top, giving rise to a sequence $x_1 > \dots > x_m$ of bumped entries. Place these $x_1 > \dots > x_m$ into

the boxes in row k in a unique way such that row k keeps weakly increasing. It is not hard to check that the final outcome is a set-valued reverse plane partition T in $\text{SVRPP}(\lambda, k, k+1)$. Moreover, by the above construction, we have $\text{di}_k(T) = T'$. This allows us to conclude the lemma. $\blacksquare$

We next define the contraction operation. For $k \geq 1$, the k -th contraction, denoted co_k , acts on the set $\text{SVRPP}(\lambda, k, k)$. Let $T \in \text{SVRPP}(\lambda, k, k)$. Equivalently, the subtableau $T_{\geq k}$ is an ordinary semistandard tableau. The image $\text{co}_k(T)$ is constructed as follows.

- Replace each box in the k -th row of T , whose entry equals the largest entry of the set immediately above it in row $k-1$, by an empty box. Then apply the jdt slides to move these empty boxes down and to the right until they exit the diagram. The slide order is from right to left, thus every original empty box can be viewed as an inside corner. This procedure can be best understood by an example as given in Figure 7 where $k = 2$.

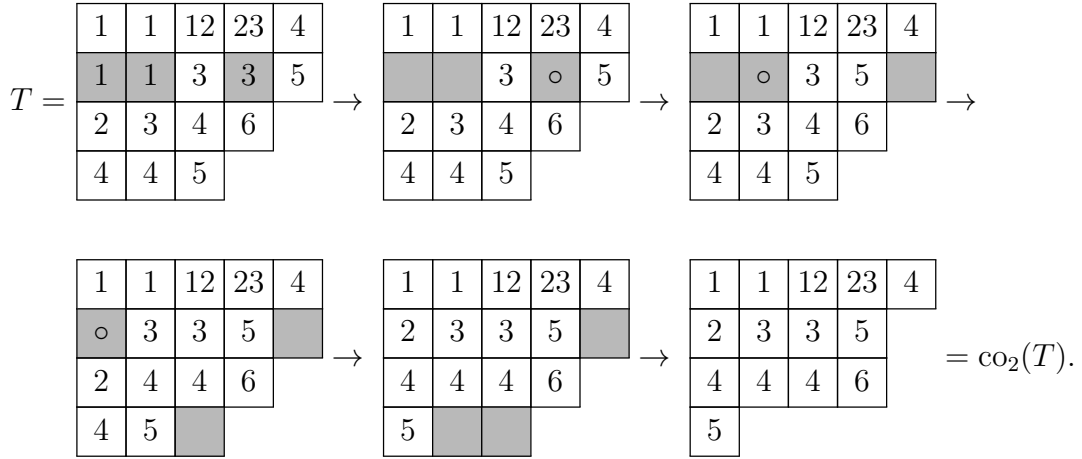


Figure 7: An illustration of the contraction algorithm.

Lemma 3.7. *Fix a shape λ . The map co_k is a bijection from $\text{SVRPP}(\lambda, k, k)$ to the union of $\text{SVRPP}(\mu, k-1, k)$, where (λ, μ) is a $(k-1)$ -falling pair.*

Proof. Let $T \in \text{SVRPP}(\lambda, k, k)$. We can forget all the entries of T in row $k-1$ except the largest one in every box and add them back after the whole process. Suppose that the empty boxes in row k are $(k, j_1), \dots, (k, j_m)$, listed from right to left. By the construction of co_k , we successively perform the jdt algorithm starting from the empty boxes in the order $(k, j_1), \dots, (k, j_m)$. A crucial observation is that the m slide paths do not intersect, distributed from upper right to lower left. Denote by T' the resulting filling. Now it is a routine task to check that T' is a set-valued reverse plane partition belonging to $\text{SVRPP}(\mu, k-1, k)$, where λ/μ is a horizontal strip (so (λ, μ) is a $(k-1)$ -falling pair).

Conversely, consider a $T' \in \text{SVRPP}(\mu, k-1, k)$, where (λ, μ) is a $(k-1)$ -falling pair. Then do the reverse jdt slides to these empty boxes in the horizontal strip λ/μ from left to right, and terminate at row k (which means the next jdt slide will move the empty box up into row $k-1$). Then fill every empty box in row k with the largest entry in the box immediately above it. It is not hard to check that the final outcome is a set-valued reverse plane partition T in $\text{SVRPP}(\lambda, k, k)$. Moreover, the above construction is indeed the reverse map of co_k . So the proof is complete. $\blacksquare$

We can now describe our dilation-contraction algorithm which transforms a set-valued reverse plane partition of shape λ to a semistandard tableau of shape μ together with an $\ell(\lambda)$ -oscillating sequence from λ to μ . Let $T \in \text{SVRPP}(\lambda)$. We first use the dilation and contraction operations to generate a sequence $(T_0, T_1, \dots, T_{2\ell(\lambda)-1})$ of set-valued reverse plane partitions. Set $l = \ell(\lambda)$ and $T_0 = T$. The tableaux T_i for $i = 1, \dots, 2l - 1$ are constructed as follows.

- Let $1 \leq i \leq 2l - 1$. If $i = 2k - 1$ is odd, then let

$$T_i = \text{di}_{l+1-k}(T_{i-1})$$

be obtained from T_{i-1} by applying the $(l+1-k)$ -th dilation. If $i = 2k$ is even, then let

$$T_i = \text{co}_{l+1-k}(T_{i-1})$$

be obtained from T_{i-1} by applying the $(l+1-k)$ -th contraction. That is,

$$T_0 \xrightarrow{\text{di}_l} T_1 \xrightarrow{\text{co}_l} T_2 \xrightarrow{\text{di}_{l-1}} T_3 \xrightarrow{\text{co}_{l-1}} T_4 \xrightarrow{\text{di}_{l-2}} \dots \xrightarrow{\text{co}_2} T_{2l-2} \xrightarrow{\text{di}_1} T_{2l-1}.$$

Figure 8 gives a concrete example to illustrate the dilation-contraction algorithm.

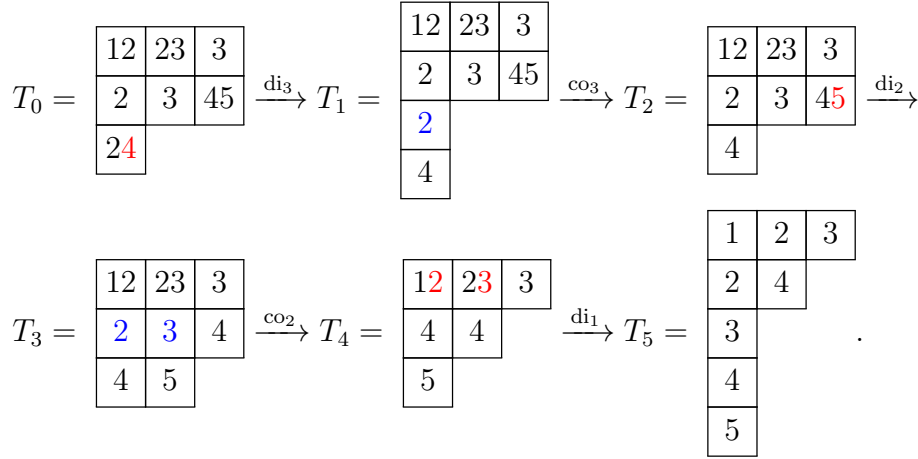


Figure 8: An illustration of the dilation-contraction algorithm.

We can now construct the map Φ in Theorem 3.4. For $T \in \text{SVRPP}(\lambda)$, let $(T_0, T_1, \dots, T_{2\ell(\lambda)-1})$ be the sequence of set-valued reverse plane partitions after applying the dilation-contraction algorithm. Notice that $T_{2\ell(\lambda)-1}$ is an ordinary semistandard tableau. For $0 \leq i \leq 2\ell(\lambda) - 1$, write λ^i for the shape of T_i . Define $\Phi(T) = (T', S)$ where $T' = T_{2\ell(\lambda)-1}$ and

$$S = (\lambda^0, \lambda^1, \dots, \lambda^{2\ell(\lambda)-1}).$$

Proof of Theorem 3.4. By Lemmas 3.6 and 3.7, it is an easy task to justify that the map Φ is indeed a bijection from $\text{SVRPP}(\lambda)$ to

$$\bigsqcup_{\mu} \text{SSYT}(\mu) \times \text{OS}^{\ell(\lambda)}(\lambda \rightarrow \mu).$$

Moreover, from the definitions of $\text{ircont}(T)$, $\text{ex}(T)$, $\text{ceq}(T)$ and $\text{wt}(S)$, we can directly check that

$$\mathbf{x}^{\text{ircont}(T)} \mathbf{w}^{\text{ex}(T)} \mathbf{t}^{\text{ceq}(T)} = \mathbf{x}^{\text{wt}(T')} \text{wt}(S).$$

This completes the proof. ■

Remark 3.8. For a partition λ , let $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots)$ be given by

$$\bar{\lambda}_i := \min\{\lambda_j + j - 1 : 1 \leq j \leq i\}.$$

By [13, Section 4],

$$(\lambda_1) \subseteq \mu \subseteq \bar{\lambda} \iff K_{\lambda, \mu}(\mathbf{t}, \mathbf{w}) \neq 0.$$

(In [13, Section 4], we defined $\bar{\lambda}^{(n)}$ and here we need to let $n \rightarrow \infty$ to obtain $\bar{\lambda}$.) The sequence $\bar{\lambda}$ is weakly decreasing, but it is not a partition because it has infinitely many nonzero parts. The notation $\mu \subseteq \bar{\lambda}$ means that $\mu_i \leq \bar{\lambda}_i$ for $i \geq 1$. Combining Corollary 3.5, we have

$$(\lambda_1) \subseteq \mu \subseteq \bar{\lambda} \iff \text{OS}^{\ell(\lambda)}(\lambda \rightarrow \mu) \neq \emptyset.$$

3.2 Distributive lattice of oscillating sequences

We show that oscillating sequences admit a natural distributive lattice structure. A crucial observation is as follows.

Lemma 3.9. If (λ, ρ) and (μ, γ) are k -rising pairs (resp., k -falling pairs), then $(\lambda \vee \mu, \rho \vee \gamma)$ and $(\lambda \wedge \mu, \rho \wedge \gamma)$ are also k -rising pairs (resp., k -falling pairs).

Proof. We only verify the case for k -rising pairs. The proof for k -falling pairs can be dealt with in an analogous manner.

For $1 \leq i \leq k$, it follows from the definition that $\lambda_i = \rho_i$ and $\mu_i = \gamma_i$, and so we have

$$(\lambda \vee \mu)_i = \max\{\lambda_i, \mu_i\} = \max\{\rho_i, \gamma_i\} = (\rho \vee \gamma)_i,$$

and

$$(\lambda \wedge \mu)_i = \min\{\lambda_i, \mu_i\} = \min\{\rho_i, \gamma_i\} = (\rho \wedge \gamma)_i.$$

It remains to show that both $(\rho \vee \gamma)/(\lambda \vee \mu)$ and $(\rho \wedge \gamma)/(\lambda \wedge \mu)$ are vertical strips.

Notice that if $\lambda \subseteq \rho$ and $\mu \subseteq \gamma$, then

$$\lambda \vee \mu \subseteq \rho \vee \gamma \quad \text{and} \quad \lambda \wedge \mu \subseteq \rho \wedge \gamma.$$

Since both ρ/λ and γ/μ are vertical strips, we see that for $j \geq 1$,

$$\rho_j - \lambda_j \leq 1, \quad \gamma_j - \mu_j \leq 1.$$

Thus, for $j \geq 1$, we deduce that

$$\begin{aligned} (\rho \vee \gamma)_j - (\lambda \vee \mu)_j &= \max\{\rho_j, \gamma_j\} - \max\{\lambda_j, \mu_j\} \\ &= \max\{\rho_j - \max\{\lambda_j, \mu_j\}, \gamma_j - \max\{\lambda_j, \mu_j\}\} \\ &\leq \max\{\rho_j - \lambda_j, \gamma_j - \mu_j\} \leq 1, \end{aligned}$$

and

$$\begin{aligned} (\rho \wedge \gamma)_j - (\lambda \wedge \mu)_j &= \min\{\rho_j, \gamma_j\} - \min\{\lambda_j, \mu_j\} \\ &= \min\{\rho_j, \gamma_j\} + \max\{-\lambda_j, -\mu_j\} \\ &= \max\{\min\{\rho_j, \gamma_j\} - \lambda_j, \min\{\rho_j, \gamma_j\} - \mu_j\} \\ &\leq \max\{\rho_j - \lambda_j, \gamma_j - \mu_j\} \leq 1, \end{aligned}$$

which implies that $(\rho \vee \gamma)/(\lambda \vee \mu)$ and $(\rho \wedge \gamma)/(\lambda \wedge \mu)$ are vertical strips. ■

Denote

$$\text{OS}^l(\lambda) = \bigcup_{\mu} \text{OS}^l(\lambda \rightarrow \mu) = \bigcup_{\mu: (\lambda_1) \subseteq \mu \subseteq \bar{\lambda}} \text{OS}^l(\lambda \rightarrow \mu),$$

where the second equality follows from Remark 3.8. We also use the notation

$$\text{OS}_k^l = \bigcup_{\lambda: \ell(\lambda) \leq l, \lambda_1 \leq k} \text{OS}^l(\lambda),$$

and

$$\text{OS}_k^{l,m} = \bigcup_{\substack{\lambda: \ell(\lambda) \leq l \\ \lambda_1 \leq k}} \bigcup_{\substack{\mu: \ell(\mu) \leq m \\ (\lambda_1) \subseteq \mu \subseteq \bar{\lambda}}} \text{OS}^l(\lambda \rightarrow \mu). \quad (3.1)$$

The following is a consequence of Lemma 3.9.

Corollary 3.10. *For two l -oscillating sequences $S = (\lambda^0, \lambda^1, \dots, \lambda^{2l-1})$ and $S' = (\mu^0, \mu^1, \dots, \mu^{2l-1})$, define*

$$S \vee S' = (\lambda^0 \vee \mu^0, \lambda^1 \vee \mu^1, \dots, \lambda^{2l-1} \vee \mu^{2l-1})$$

and

$$S \wedge S' = (\lambda^0 \wedge \mu^0, \lambda^1 \wedge \mu^1, \dots, \lambda^{2l-1} \wedge \mu^{2l-1}).$$

Then $S \vee S'$ and $S \wedge S'$ are also l -oscillating sequences. With the above join and meet, the sets $\text{OS}^l(\lambda \rightarrow \mu)$, $\text{OS}^l(\lambda)$, OS_k^l and $\text{OS}_k^{l,m}$ are all distributive lattices. In particular, the first and the last are finite distributive lattices.

4 Multivariate Reuter–Lovász–Saks inequality

In this section, we present a multivariate Reuter–Lovász–Saks type inequality which is another ingredient in the proof of Theorem 1.5. For the history of the classical Reuter–Lovász–Saks inequality, we refer the reader to [4] for an overview.

Throughout this section, let $(L, \wedge, \vee)$ be a *finite* distributive lattice. Fix variables $\mathbf{q} = (q_1, \dots, q_N)$, and *modular functions* $\mathbf{r} = (r_1, \dots, r_N)$, that is, for $1 \leq i \leq N$, $r_i : L \rightarrow \mathbb{Z}_{\geq 0}$ satisfies

$$r_i(x) + r_i(y) = r_i(x \vee y) + r_i(x \wedge y) \quad \text{for } x, y \in L. \quad (4.1)$$

For $x \in L$, write $\mathbf{q}^{\mathbf{r}(x)} := q_1^{r_1(x)} \cdots q_N^{r_N(x)}$. For a function $\rho : L \rightarrow \mathbb{R}_{\geq 0}$ and subset $X \subseteq L$, define

$$\rho_{\langle \mathbf{q}, \mathbf{r} \rangle}(X) := \sum_{x \in X} \rho(x) \mathbf{q}^{\mathbf{r}(x)} \in \mathbb{R}_{\geq 0}[q_1, \dots, q_N].$$

For two polynomials f, g in $\mathbf{q}$, write $f \leq^{\mathbf{q}} g$ if

$$g - f \in \mathbb{R}_{\geq 0}[\mathbf{q}].$$

For two subsets $X, Y \subseteq L$, let $X \vee Y = \{x \vee y : x \in X, y \in Y\}$ and $X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$. The following is a multivariate Reuter–Lovász–Saks type inequality.

Theorem 4.1 (multivariate RLS type inequality). *Let $(L, \wedge, \vee)$ be a finite distributive lattice, and let $r_1, \dots, r_N : L \rightarrow \mathbb{Z}_{\geq 0}$ be modular functions satisfying (4.1). Suppose that for every subset $H \subseteq [n] := \{1, \dots, n\}$, we are given subsets*

$$A_H, B_H, C_H, D_H \subseteq L$$

such that for any $I, J \subseteq [n]$, $A_I \vee B_J \subseteq C_{I \cup J}$ and $A_I \wedge B_J \subseteq D_{I \cap J}$. For a subset $X \subseteq L$, define

$$X(\mathbf{q}) = \sum_{x \in X} \mathbf{q}^{\mathbf{r}(x)}.$$

Then

$$\sum_{H \subseteq [n]} A_H(\mathbf{q}) B_{\overline{H}}(\mathbf{q}) \leq^{\mathbf{q}} \sum_{H \subseteq [n]} C_H(\mathbf{q}) D_{\overline{H}}(\mathbf{q}), \quad \text{where } \overline{H} = [n] \setminus H.$$

To give a proof of Theorem 4.1, we need the multivariate Ahlswede–Daykin inequality due to Chan and Pak [5, Theorem 6.1].

Theorem 4.2 ([5, Theorem 6.1]). *Let $(L, \wedge, \vee)$ be a finite distributive lattice, and let $\alpha, \beta, \rho, \delta : L \rightarrow \mathbb{R}_{\geq 0}$ be nonnegative functions on L . Suppose that*

$$\alpha(x) \cdot \beta(y) \leq \rho(x \vee y) \cdot \delta(x \wedge y) \quad \text{for } x, y \in L.$$

Then, for $X, Y \subseteq L$, we have

$$\alpha_{\langle \mathbf{q}, \mathbf{r} \rangle}(X) \cdot \beta_{\langle \mathbf{q}, \mathbf{r} \rangle}(Y) \leq^{\mathbf{q}} \rho_{\langle \mathbf{q}, \mathbf{r} \rangle}(X \vee Y) \cdot \delta_{\langle \mathbf{q}, \mathbf{r} \rangle}(X \wedge Y).$$

We now give a proof of Theorem 4.1.

Proof of Theorem 4.1. We first encode a subset $H \subseteq [n]$ by extra variables $\mathbf{z} = (z_1, \dots, z_n)$, that is, write

$$\mathbf{z}^H = \prod_{i \in H} z_i.$$

Consider the product lattice

$$\tilde{L} = L \times 2^{[n]},$$

with join and meet given by

$$(x, H) \vee (y, J) = (x \vee y, H \cup J), \quad (x, H) \wedge (y, J) = (x \wedge y, H \cap J).$$

With the above operations, $\tilde{L}$ is a finite distributive lattice.

We extend the modular functions on L to $\tilde{L}$ by setting

$$\tilde{r}_i(x, H) = r_i(x), \quad 1 \leq i \leq N.$$

Moreover, we introduce Boolean modular functions: for $1 \leq j \leq n$,

$$\varepsilon_j(x, H) = \begin{cases} 1, & j \in H, \\ 0, & j \notin H. \end{cases}$$

Indeed, one has

$$\varepsilon_j(x, H) + \varepsilon_j(y, J) = \varepsilon_j(x \vee y, H \cup J) + \varepsilon_j(x \wedge y, H \cap J),$$

because for each fixed $j \in [n]$,

$$\mathbf{1}_{j \in H} + \mathbf{1}_{j \in J} = \mathbf{1}_{j \in H \cup J} + \mathbf{1}_{j \in H \cap J}.$$

Now we see that $\tilde{r}_1, \dots, \tilde{r}_N, \varepsilon_1, \dots, \varepsilon_n$ are modular functions on $\tilde{L}$.

Define four subsets of $\tilde{L}$ by

$$\begin{aligned}\mathcal{A} &= \{(a, H) : H \subseteq [n], a \in A_H\}, \\ \mathcal{B} &= \{(b, J) : J \subseteq [n], b \in B_J\}, \\ \mathcal{C} &= \{(c, K) : K \subseteq [n], c \in C_K\}, \\ \mathcal{D} &= \{(d, M) : M \subseteq [n], d \in D_M\}.\end{aligned}$$

We claim that

$$\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{C}, \quad \mathcal{A} \wedge \mathcal{B} \subseteq \mathcal{D}.$$

This can be seen as follows. If $(a, H) \in \mathcal{A}$ and $(b, J) \in \mathcal{B}$, then $a \in A_H$ and $b \in B_J$. By the assumption in the theorem, we have

$$a \vee b \in C_{H \cup J}, \quad a \wedge b \in D_{H \cap J}.$$

Therefore

$$(a, H) \vee (b, J) = (a \vee b, H \cup J) \in \mathcal{C},$$

and

$$(a, H) \wedge (b, J) = (a \wedge b, H \cap J) \in \mathcal{D}.$$

This verifies the claim.

Applying Theorem 4.2, along with the above claim, to $\tilde{L}$ with $\alpha = \beta = \rho = \delta = 1$ and with the modular functions $\tilde{r}_1, \dots, \tilde{r}_N, \varepsilon_1, \dots, \varepsilon_n$, we deduce that

$$\begin{aligned}\left(\sum_{(a,H) \in \mathcal{A}} \mathbf{q}^{\mathbf{r}(a)} \mathbf{z}^H \right) \left(\sum_{(b,J) \in \mathcal{B}} \mathbf{q}^{\mathbf{r}(b)} \mathbf{z}^J \right) &\leq^{\mathbf{q}, \mathbf{z}} \left(\sum_{(c,K) \in \mathcal{A} \vee \mathcal{B}} \mathbf{q}^{\mathbf{r}(c)} \mathbf{z}^K \right) \left(\sum_{(d,M) \in \mathcal{A} \wedge \mathcal{B}} \mathbf{q}^{\mathbf{r}(d)} \mathbf{z}^M \right) \\ &\leq^{\mathbf{q}, \mathbf{z}} \left(\sum_{(c,K) \in \mathcal{C}} \mathbf{q}^{\mathbf{r}(c)} \mathbf{z}^K \right) \left(\sum_{(d,M) \in \mathcal{D}} \mathbf{q}^{\mathbf{r}(d)} \mathbf{z}^M \right).\end{aligned}$$

Equivalently,

$$\left(\sum_{H \subseteq [n]} A_H(\mathbf{q}) \mathbf{z}^H \right) \left(\sum_{J \subseteq [n]} B_J(\mathbf{q}) \mathbf{z}^J \right) \leq^{\mathbf{q}, \mathbf{z}} \left(\sum_{K \subseteq [n]} C_K(\mathbf{q}) \mathbf{z}^K \right) \left(\sum_{M \subseteq [n]} D_M(\mathbf{q}) \mathbf{z}^M \right).$$

We now take the coefficients of $\mathbf{z}^{[n]} = z_1 z_2 \cdots z_n$ on both sides. Since each $\mathbf{z}^H$ is squarefree, the coefficient of $\mathbf{z}^{[n]}$ on the left-hand side is obtained precisely from pairs (H, J) satisfying $H \cap J = \emptyset, H \cup J = [n]$. Equivalently, $J = \overline{H}$. Hence

$$[\mathbf{z}^{[n]}] \left(\sum_{H \subseteq [n]} A_H(\mathbf{q}) \mathbf{z}^H \right) \left(\sum_{J \subseteq [n]} B_J(\mathbf{q}) \mathbf{z}^J \right) = \sum_{H \subseteq [n]} A_H(\mathbf{q}) B_{\overline{H}}(\mathbf{q}).$$

Similarly,

$$[\mathbf{z}^{[n]}] \left(\sum_{K \subseteq [n]} C_K(\mathbf{q}) \mathbf{z}^K \right) \left(\sum_{M \subseteq [n]} D_M(\mathbf{q}) \mathbf{z}^M \right) = \sum_{K \subseteq [n]} C_K(\mathbf{q}) D_{\overline{K}}(\mathbf{q}).$$

Note that the inequality before taking coefficients is coefficientwise in both $\mathbf{q}$ and $\mathbf{z}$. Hence taking the coefficient of $\mathbf{z}^{[n]}$ preserves coefficientwise nonnegativity in $\mathbf{q}$. This implies that

$$\sum_{H \subseteq [n]} A_H(\mathbf{q}) B_{\overline{H}}(\mathbf{q}) \leq^{\mathbf{q}} \sum_{H \subseteq [n]} C_H(\mathbf{q}) D_{\overline{H}}(\mathbf{q}),$$

and so the proof is complete. ■

5 Schur orchestra inequality

In this section, we recall the Schur orchestra inequality developed by Chan, Chen, Pak and Soskin [4]. We then present a variation of this inequality which will be needed in the proof of Theorem 1.5.

Let

$$\mathcal{E} = \{1, \dots, \ell\} \times \{0\}, \quad \mathcal{F} = \{1, \dots, \ell\} \times \{1\}$$

be two disjoint copies of $[\ell]$. Denote by $\mathcal{E} \sqcup \mathcal{F}$ their disjoint union. Let

$$O : 2^{\mathcal{E} \sqcup \mathcal{F}} \longrightarrow \mathbb{R}$$

be a real-valued function on the Boolean lattice of subsets of $\mathcal{E} \sqcup \mathcal{F}$. We say that O satisfies the orchestra inequalities if for every set partition $\mathcal{E} \sqcup \mathcal{F} = B_1 \sqcup \dots \sqcup B_t$,

$$\sum_{H \subseteq [t]} O\left(\bigsqcup_{h \in H} B_h\right) \geq 0.$$

Suppose that

$$\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}^\ell.$$

For a subset $E \subseteq \mathcal{E}$, define $\lambda(E) = (\lambda_1(E), \dots, \lambda_\ell(E))$ by

$$\lambda_i(E) = \begin{cases} \alpha_i + \gamma_i, & i \in E, \\ \alpha_i, & i \notin E, \end{cases} \quad (5.1)$$

where we simply use $i \in E$ to represent $(i, 0) \in E$. Similarly, for a subset $F \subseteq \mathcal{F}$, define $\mu(F) = (\mu_1(F), \dots, \mu_\ell(F))$ by

$$\mu_j(F) = \begin{cases} \beta_j + \delta_j, & j \in F, \\ \beta_j, & j \notin F. \end{cases}$$

For $E \subseteq \mathcal{E}$ and $F \subseteq \mathcal{F}$, write $\overline{E} = \mathcal{E} \setminus E$ and $\overline{F} = \mathcal{F} \setminus F$. We also adopt the following convention: if λ or μ is not a partition, or if $\mu \not\leq \lambda$, then $s_{\lambda/\mu} = 0$.

Theorem 5.1 (Schur orchestra inequality [4, Theorem 5.2]). *Let*

$$O : 2^{\mathcal{E} \sqcup \mathcal{F}} \longrightarrow \mathbb{R}$$

be a function satisfying the orchestra inequalities. Then

$$\sum_{\substack{E \subseteq \mathcal{E} \\ F \subseteq \mathcal{F}}} O(E \sqcup F) s_{\lambda(E)/\mu(F)}(\mathbf{x}) s_{\lambda(\overline{E})/\mu(\overline{F})}(\mathbf{x}) \geq_s 0.$$

What we require is a variation of the Schur orchestra inequality.

Theorem 5.2. *Let $I = [\ell]$, and let $O : 2^I \longrightarrow \mathbb{R}$ be a function satisfying the orchestra inequalities: for every set partition $I = B_1 \sqcup \dots \sqcup B_t$,*

$$\sum_{H \subseteq [t]} O\left(\bigcup_{h \in H} B_h\right) \geq 0.$$

Let $\alpha, \gamma \in \mathbb{Z}_{\geq 0}^\ell$. For $E \subseteq I$, define $\lambda(E)$ as in (5.1). Then

$$\sum_{E \subseteq I} O(E) s_{\lambda(E)}(\mathbf{x}) s_{\lambda(I \setminus E)}(\mathbf{x}) \geq_s 0.$$

Proof. Let

$$\mathcal{E} = I \times \{0\}, \quad \mathcal{F} = I \times \{1\}$$

be two disjoint copies of I . For $A \subseteq \mathcal{E}$, define the projection

$$\pi(A) = \{i \in I : (i, 0) \in A\}.$$

Construct a function

$$\tilde{O} : 2^{\mathcal{E} \sqcup \mathcal{F}} \longrightarrow \mathbb{R}$$

by

$$\tilde{O}(A \sqcup B) = 2^{-\#I} O(\pi(A)), \quad A \subseteq \mathcal{E}, B \subseteq \mathcal{F}.$$

Thus $\tilde{O}$ depends only on the part lying in the first copy $\mathcal{E}$.

We first prove that $\tilde{O}$ satisfies the orchestra inequalities on $\mathcal{E} \sqcup \mathcal{F}$. Let

$$\mathcal{E} \sqcup \mathcal{F} = C_1 \sqcup \cdots \sqcup C_t$$

be an arbitrary set partition. For each $j \in [t]$, define

$$D_j = \pi(C_j \cap \mathcal{E}) \subseteq I.$$

The nonempty sets among $D_1, \dots, D_t$ are pairwise disjoint and have union I . Let

$$M = \{j \in [t] : D_j \neq \emptyset\}, \quad m = \#M.$$

Then

$$I = \bigsqcup_{j \in M} D_j.$$

For every $H \subseteq [t]$, we have

$$\pi \left(\left(\bigsqcup_{h \in H} C_h \right) \cap \mathcal{E} \right) = \bigcup_{h \in H} D_h = \bigcup_{h \in H \cap M} D_h.$$

Hence

$$\begin{aligned} \sum_{H \subseteq [t]} \tilde{O} \left(\bigsqcup_{h \in H} C_h \right) &= 2^{-\#I} \sum_{H \subseteq [t]} O \left(\bigcup_{h \in H} D_h \right) \\ &= 2^{-\#I} 2^{t-m} \sum_{H' \subseteq M} O \left(\bigcup_{h \in H'} D_h \right). \end{aligned} \quad (5.2)$$

Indeed, after fixing $H' = H \cap M$, the remaining indices in $[t] \setminus M$ can be chosen arbitrarily, and they do not change the union $\bigcup_{h \in H} D_h$, since $D_j = \emptyset$ for $j \notin M$.

Since the nonempty D_j 's form a set partition of I , the sum in (5.2) is nonnegative by the orchestra inequalities satisfied by O . Therefore $\tilde{O}$ satisfies the orchestra inequalities on $\mathcal{E} \sqcup \mathcal{F}$.

We now apply Theorem 5.1 to $\tilde{O}$ where we set $\beta = \delta = \mathbf{0}$. For $A \subseteq \mathcal{E}$ and $B \subseteq \mathcal{F}$, we see that

$$s_{\lambda(A)/\mu(B)}(\mathbf{x}) = s_{\lambda(\pi(A))}(\mathbf{x}),$$

and

$$s_{\lambda(\mathcal{E} \setminus A)/\mu(\mathcal{F} \setminus B)}(\mathbf{x}) = s_{\lambda(I \setminus \pi(A))}(\mathbf{x}).$$

Theorem 5.1 therefore gives

$$\begin{aligned}
0 &\leq_s \sum_{A \subseteq \mathcal{E}} \sum_{B \subseteq \mathcal{F}} \tilde{O}(A \sqcup B) s_{\lambda(\pi(A))}(\mathbf{x}) s_{\lambda(I \setminus \pi(A))}(\mathbf{x}) \\
&= \sum_{A \subseteq \mathcal{E}} \sum_{B \subseteq \mathcal{F}} 2^{-\#I} O(\pi(A)) s_{\lambda(\pi(A))}(\mathbf{x}) s_{\lambda(I \setminus \pi(A))}(\mathbf{x}).
\end{aligned} \tag{5.3}$$

For a fixed $A \subseteq \mathcal{E}$, the summand is independent of $B \subseteq \mathcal{F}$, and there are $2^{\#I}$ choices of B . Hence the normalizing factor $2^{-\#I}$ cancels the sum over B . Since $A \mapsto \pi(A)$ is a bijection from subsets of $\mathcal{E}$ to subsets of I , (5.3) becomes

$$\sum_{E \subseteq I} O(E) s_{\lambda(E)}(\mathbf{x}) s_{\lambda(I \setminus E)}(\mathbf{x}),$$

as desired. ■

6 Proof of Theorem 1.5

We will finish the proof of Theorem 1.5 in this section. The main ingredients include Corollaries 3.5 and 3.10, Theorem 4.1, as well as Theorem 5.2.

Theorem 6.1 (=Theorem 1.5). *For partitions λ, μ , we have*

$$H_{\lambda}(\mathbf{x}; \mathbf{t}, \mathbf{w}) H_{\mu}(\mathbf{x}; \mathbf{t}, \mathbf{w}) \leq_s^{\mathbf{t}, \mathbf{w}} H_{\lambda \vee \mu}(\mathbf{x}; \mathbf{t}, \mathbf{w}) H_{\lambda \wedge \mu}(\mathbf{x}; \mathbf{t}, \mathbf{w}).$$

Proof. Let $l = \max\{\ell(\lambda), \ell(\mu)\}$. For an l -oscillating sequence

$$S = (\lambda^0, \lambda^1, \dots, \lambda^{2l-1}),$$

define $\mathbf{r}(S) = (r_1(S), \dots, r_{2l-1}(S))$ by setting

$$r_{2i-1}(S) = |\lambda^{2i-1} / \lambda^{2i-2}| \text{ for } 1 \leq i \leq l, \quad r_{2j}(S) = |\lambda^{2j-1} / \lambda^{2j}| \text{ for } 1 \leq j \leq l-1.$$

For two l -oscillating sequences S and S' , it is readily checked that for $1 \leq i \leq 2l-1$,

$$r_i(S) + r_i(S') = r_i(S \vee S') + r_i(S \wedge S'). \tag{6.1}$$

Write $\mathbf{q} = (q_1, \dots, q_{2l-1})$. Then

$$\mathbf{q}^{\mathbf{r}(S)} = \prod_{1 \leq i \leq l} q_{2i-1}^{r_{2i-1}(S)} \prod_{1 \leq j \leq l-1} q_{2j}^{r_{2j}(S)}.$$

For partitions λ, α , define

$$K_{\lambda, \alpha}(\mathbf{q}) = \sum_{S \in \text{OS}^l(\lambda \rightarrow \alpha)} \mathbf{q}^{\mathbf{r}(S)}.$$

For $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{2l-1}$, set

$$F_{\alpha}^{\mathbf{a}}(\beta) = \#\{S \in \text{OS}^l(\alpha \rightarrow \beta) : \mathbf{r}(S) = \mathbf{a}\}.$$

Thus

$$K_{\alpha, \beta}(\mathbf{q}) = \sum_{\mathbf{a}} F_{\alpha}^{\mathbf{a}}(\beta) \mathbf{q}^{\mathbf{a}}.$$

Let

$$H_\lambda(\mathbf{x}; \mathbf{q}) = \sum_{\alpha} K_{\lambda, \alpha}(\mathbf{q}) s_{\alpha}(\mathbf{x}),$$

which, together with Corollary 3.5, specializes to $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$ after the substitutions

$$q_{2i-1} = w_{l-i+1} \ (1 \leq i \leq l), \quad q_{2j} = t_{l-j} \ (1 \leq j \leq l-1). \quad (6.2)$$

Remark. Strictly speaking, in Corollary 3.5, we require $l = \ell(\lambda)$. In the case when $l > \ell(\lambda)$, we notice that the first $2(l - \ell(\lambda))$ shapes in any oscillating sequence in $\text{OS}^l(\lambda \rightarrow \alpha)$ are equal, and so there is a bijection from $\text{OS}^l(\lambda \rightarrow \alpha)$ to $\text{OS}^{\ell(\lambda)}(\lambda \rightarrow \alpha)$ by ignoring the initial $2(l - \ell(\lambda))$ shapes. In particular, after the substitutions in (6.2), $H_\lambda(\mathbf{x}; \mathbf{q})$ still specializes to $H_\lambda(\mathbf{x}; \mathbf{t}, \mathbf{w})$.

For $\zeta \in \mathbb{Z}_{\geq 0}^{2l-1}$, denote

$$L_\zeta(\nu, \gamma) = \sum_{\mathbf{a}+\mathbf{b}=\zeta} F_\lambda^{\mathbf{a}}(\nu) F_\mu^{\mathbf{b}}(\gamma),$$

and

$$R_\zeta(\nu, \gamma) = \sum_{\mathbf{a}+\mathbf{b}=\zeta} F_{\lambda \vee \mu}^{\mathbf{a}}(\nu) F_{\lambda \wedge \mu}^{\mathbf{b}}(\gamma).$$

Then the coefficient of $\mathbf{q}^\zeta$ in

$$H_{\lambda \vee \mu}(\mathbf{x}; \mathbf{q}) H_{\lambda \wedge \mu}(\mathbf{x}; \mathbf{q}) - H_\lambda(\mathbf{x}; \mathbf{q}) H_\mu(\mathbf{x}; \mathbf{q})$$

is

$$\Delta_\zeta = \sum_{\nu, \gamma} (R_\zeta(\nu, \gamma) - L_\zeta(\nu, \gamma)) s_\nu s_\gamma. \quad (6.3)$$

Therefore, it is enough to prove that for every fixed ζ , Δ_ζ is Schur positive. The proof will be divided into three steps.

Step 1. We first explain that the sum in (6.3) is finite for each fixed $\zeta \in \mathbb{Z}_{\geq 0}^{2l-1}$. It follows from the definition that if $F_\lambda^{\mathbf{a}}(\nu) \neq 0$, then there exists

$$S = (\lambda^0, \lambda^1, \dots, \lambda^{2l-1}) \in \text{OS}^l(\lambda \rightarrow \nu)$$

with $\mathbf{r}(S) = \mathbf{a}$. Recall that

$$a_{2i-1} = |\lambda^{2i-1} / \lambda^{2i-2}| \ (1 \leq i \leq l), \quad a_{2j} = |\lambda^{2j-1} / \lambda^{2j}| \ (1 \leq j \leq l-1).$$

Hence

$$|\nu| = |\lambda| + \sum_{i=1}^l a_{2i-1} - \sum_{j=1}^{l-1} a_{2j}.$$

Thus, for fixed λ and $\mathbf{a}$, the endpoint shape ν must have a fixed size, and so there are finitely many such partitions ν .

For fixed ζ , there are only finitely many decompositions $\mathbf{a} + \mathbf{b} = \zeta$. It follows that $L_\zeta(\nu, \gamma)$ and $R_\zeta(\nu, \gamma)$ are nonzero for only finitely many pairs (ν, γ) .

The following two steps will borrow some ideas in the proofs of [4, Theorem 1.7] and [4, Theorem 8.1].

Step 2. We now group the summands of Δ_ζ in (6.3) according to their join and meet. Fix a pair of partitions $\sigma \subseteq \rho$, and consider the fiber

$$\mathcal{F}(\rho, \sigma) = \{(\nu, \gamma) : \nu \vee \gamma = \rho, \ \nu \wedge \gamma = \sigma\}.$$

Note that for two pairs of partitions $\sigma^1 \subseteq \rho^1$ and $\sigma^2 \subseteq \rho^2$, if $(\sigma^1, \rho^1) \neq (\sigma^2, \rho^2)$, then

$$\mathcal{F}(\rho^1, \sigma^1) \cap \mathcal{F}(\rho^2, \sigma^2) = \emptyset.$$

Hence (6.3) can be split into disjoint summands according to $\mathcal{F}(\rho, \sigma)$.

Let $l' = \max\{\ell(\rho), \ell(\sigma)\}$, and set

$$I = \{1, 2, \dots, l'\}. \quad (6.4)$$

For $E \subseteq I$, define two vectors ν_E and $\gamma_E = \nu_{I \setminus E}$ in $\mathbb{Z}_{\geq 0}^{l'}$ by

$$\nu_E(i) = \begin{cases} \rho_i, & i \in E, \\ \sigma_i, & i \notin E, \end{cases} \quad \gamma_E(i) = \begin{cases} \sigma_i, & i \in E, \\ \rho_i, & i \notin E. \end{cases}$$

Observe that every pair (ν, γ) satisfying $\nu \vee \gamma = \rho$ and $\nu \wedge \gamma = \sigma$ is obtained in exactly $2^{\#\{i: \rho_i = \sigma_i\}}$ ways by taking E satisfying that

$$\{i \in I : \nu_i = \rho_i, \rho_i \neq \sigma_i\} \subseteq E \subseteq \{i \in I : \nu_i = \rho_i\}.$$

Hence the contribution of $\mathcal{F}(\rho, \sigma)$ is

$$\begin{aligned} \Delta_{\zeta; \rho, \sigma} &= \sum_{(\nu, \gamma) \in \mathcal{F}(\rho, \sigma)} (R_{\zeta}(\nu, \gamma) - L_{\zeta}(\nu, \gamma)) s_{\nu} s_{\gamma} = \sum_{E \subseteq I} O_{\zeta}(E) s_{\nu_E} s_{\gamma_E} \\ &= \sum_{E \subseteq I} O_{\zeta}(E) s_{\nu_E} s_{\nu_{I \setminus E}}, \end{aligned} \quad (6.5)$$

where

$$O_{\zeta}(E) = 2^{-\#\{i: \rho_i = \sigma_i\}} (R_{\zeta}(\nu_E, \gamma_E) - L_{\zeta}(\nu_E, \gamma_E)).$$

Here, if ν_E or γ_E is not a partition, the corresponding terms are understood as zero. Now, to confirm the Schur positivity of Δ_{ζ} , it is enough to show that each $\Delta_{\zeta; \rho, \sigma}$ is Schur positive.

Step 3. We apply Theorem 5.2 to prove the Schur positivity of $\Delta_{\zeta; \rho, \sigma}$ in (6.5). To match the notation of Theorem 5.2, take $\alpha = \sigma$ and $\gamma = \rho - \sigma$. Notice that $\lambda(E) = \nu_E$ and $\lambda(I \setminus E) = \nu_{I \setminus E} = \gamma_E$. It suffices to verify that O_{ζ} satisfies the orchestra inequalities required in Theorem 5.2. Fix any set partition $B_1 \sqcup \dots \sqcup B_t$ of the set I in (6.4). For $H \subseteq [t]$, let

$$E_H = \bigcup_{h \in H} B_h, \quad \nu_H = \nu_{E_H}, \quad \gamma_H = \gamma_{E_H} = \nu_{E_H^c}.$$

The orchestra inequality we need to prove is

$$\sum_{H \subseteq [t]} O_{\zeta}(E_H) \geq 0,$$

which is equivalent to (both sides multiplied by the factor $2^{\#\{i: \rho_i = \sigma_i\}}$)

$$\sum_{H \subseteq [t]} [\mathbf{q}^{\zeta}] K_{\lambda \vee \mu, \nu_H}(\mathbf{q}) K_{\lambda \wedge \mu, \gamma_H}(\mathbf{q}) \geq \sum_{H \subseteq [t]} [\mathbf{q}^{\zeta}] K_{\lambda, \nu_H}(\mathbf{q}) K_{\mu, \gamma_H}(\mathbf{q}).$$

We will prove a stronger version

$$\sum_{H \subseteq [t]} K_{\lambda, \nu_H}(\mathbf{q}) K_{\mu, \gamma_H}(\mathbf{q}) \leq^{\mathbf{q}} \sum_{H \subseteq [t]} K_{\lambda \vee \mu, \nu_H}(\mathbf{q}) K_{\lambda \wedge \mu, \gamma_H}(\mathbf{q}). \quad (6.6)$$

We use Theorem 4.1 to give a proof of (6.6). By Step 1, we see that there exists an integer m such that every partition ν or γ appearing in a nonzero term of Δ_ζ satisfies $\ell(\nu), \ell(\gamma) \leq m$. Set $k = \max\{\lambda_1, \mu_1\}$. Consider the finite distributive lattice $\text{OS}_k^{l,m}$, as defined in (3.1). For each $H \subseteq [t]$, define subsets of $\text{OS}_k^{l,m}$ by

$$\begin{aligned} A_H &= \text{OS}^l(\lambda \rightarrow \nu_H), & B_H &= \text{OS}^l(\mu \rightarrow \nu_H), \\ C_H &= \text{OS}^l(\lambda \vee \mu \rightarrow \nu_H), & D_H &= \text{OS}^l(\lambda \wedge \mu \rightarrow \nu_H). \end{aligned}$$

Here when ν_H is not a valid partition, the above sets are understood as empty sets. Then

$$\sum_{S \in A_H} \mathbf{q}^{\mathbf{r}(S)} = K_{\lambda, \nu_H}(\mathbf{q}), \quad \sum_{T \in B_H} \mathbf{q}^{\mathbf{r}(T)} = K_{\mu, \nu_H}(\mathbf{q}),$$

and similarly for C_H, D_H .

It remains to verify the hypotheses in Theorem 4.1. Take $S \in A_H$ and $S' \in B_J$. The initial shapes of $S \vee S'$ and $S \wedge S'$ are $\lambda \vee \mu$ and $\lambda \wedge \mu$, and their final shapes are $\nu_H \vee \nu_J$ and $\nu_H \wedge \nu_J$. By the construction of ν_H , one has

$$\nu_H \vee \nu_J = \nu_{H \cup J}, \quad \nu_H \wedge \nu_J = \nu_{H \cap J}.$$

Therefore

$$A_H \vee B_J \subseteq C_{H \cup J}, \quad A_H \wedge B_J \subseteq D_{H \cap J}.$$

Moreover, we see from (6.1) that $\mathbf{r} = (r_1, \dots, r_{2l-1})$ are modular functions. Applying Theorem 4.1 to these four families, we have

$$\sum_{H \subseteq [t]} \left(\sum_{S \in A_H} \mathbf{q}^{\mathbf{r}(S)} \right) \left(\sum_{T \in B_{\overline{H}}} \mathbf{q}^{\mathbf{r}(T)} \right) \leq^{\mathbf{q}} \sum_{H \subseteq [t]} \left(\sum_{U \in C_H} \mathbf{q}^{\mathbf{r}(U)} \right) \left(\sum_{V \in D_{\overline{H}}} \mathbf{q}^{\mathbf{r}(V)} \right),$$

which, along with $\gamma_H = \nu_{\overline{H}}$, is exactly (6.6).

Collecting the above analysis in Steps 1, 2 and 3, we conclude that for every fixed ζ , Δ_ζ in (6.3) is Schur positive. This enables us to complete the proof. $\blacksquare$

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