# On graphs whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity

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**Abstract**

The equivalence of group connectivity for non-homogeneous groups with a same order has been concerned since Jaeger, Linial, Payan and Tarsi introduced this concept in [J. Combin. Theory Ser. B, 56 (1992) 165-182]. Hušek, Mohelníková and Šámal in [J. Graph Theory, 93 (2020) 317-327] showed that $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity are not equivalent by finding counterexamples with a computer-assisted proof, and they asked whether one can find a proof that does not use computers. Langhede and Thomassen [European J. Combin., (2023) 103816] provide a computer-free proof to show that there exist 3-edge-connected, $\mathbb{Z}_2^2$-connected, and non-$\mathbb{Z}_4$-connected graphs. In this paper, we construct 3-edge-connected graphs which are $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected in which we prove those properties without any involvement of computers. These two results together answer the question proposed by Hušek et al. about computer-free proofs on the non-equivalence of $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity. In addition, by using both theoretical reductions and computer searching we find the smallest graph whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity. This smallest graph (in terms of order and size) is unique, which has 10 vertices and 14 edges.

**Keywords:** group connectivity, group flows, nowhere-zero flows

## 1 Introduction

Finitely generated Abelian (additive) groups and finite loopless graphs which may contain parallel edges are considered in this study. Denote by $\Gamma$ an Abelian group with order $|\Gamma| \geq 3$ and denote by 0 the additive identity of $\Gamma$ throughout the paper. We follow [1] for undefined terms.

Let $G$ be a graph and $D$ an orientation of $G$. The set of arcs in $D$ with tail (head, resp.) as $v$ is denoted by $E^+_D(v)$ ($E^-_D(v)$, resp.). For any mapping $f : E(G) \to \Gamma$, define

$$\partial_D f(v) = \sum_{e \in E^+_D(v)} f(e) - \sum_{e \in E^-_D(v)} f(e)$$

for any vertex $v$ in $V(G)$. The pair $(D, f)$ is called a $\Gamma$-*flow* (or *flow*) of $G$ if $\partial_D f(v) = 0$ for each $v \in V(G)$. We sometimes abbreviate $(D, f)$ and $\partial_D f$ as $f$ and $\partial f$ for convenience, if the orientation $D$ is understood in context. If $\Gamma = \mathbb{Z}$, then a $\Gamma$-flow is called an *integer flow*. Furthermore, if an integer flow $(D, f)$ satisfies $|f(e)| \leq k$ for each edge $e$ in $E(G)$, then it is called a $k$-*flow*. A flow $(D, f)$ is *nowhere-zero* if $f(e) \neq 0$ for any $e \in E(G)$.

In 1992, Jaeger, Linial, Payan and Tarsi [7] extended the concept of $\Gamma$-flow to $\Gamma$-connectivity. Let $G$ be a graph. We call $\beta : V(G) \to \Gamma$ a zero-sum boundary of $G$ on $\Gamma$ if $\sum_{v \in V(G)} \beta(v) = 0$. Denote $Z(G, \Gamma)$ the collection of all zero-sum boundaries of $G$ on $\Gamma$. A flow with $f : E(G) \to \Gamma - \{0\}$ such that $\partial f = \beta$ is called a $(\Gamma, \beta)$-flow of $G$. If for any $\beta \in Z(G, \Gamma)$, there is a $(\Gamma, \beta)$-flow of $G$, then $G$ is called $\Gamma$-connected.

Tutte [16] proved that the existence of $\Gamma$-flow only depends on the order of $\Gamma$, that is, a graph $G$ has a nowhere-zero $\Gamma$-flow with $|\Gamma| = k$ if and only if $G$ has a nowhere-zero $k$-flow. Jaeger et al. [7] asked if there is a similar property for $\Gamma$-connectivity, i.e. whether $G$ is $\Gamma_1$-connected if and only if $G$ is $\Gamma_2$-connected for any two Abelian groups $\Gamma_1, \Gamma_2$ with $|\Gamma_1| = |\Gamma_2|$. They further remarked that it was even unknown for groups with order 4, which is the smallest number for the existence of non-homogeneous groups with a same order. In 2020, Hušek, Mohelníková and Šámal [6] proved that $\mathbb{Z}_4$-connectivity does not imply $\mathbb{Z}_2^2$-connectivity, neither vice versa, using a computer-aided method.

**Theorem 1.1** [6] *There exists a graph that is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected. In addition, there exists a $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected graph.*

Actually, Jaeger et al. [7] proved that every 3-edge-connected graph is $\Gamma$-connected if $|\Gamma| \geq 6$. So the equivalence of group connectivity for 3-edge-connected graphs only leaves the case in which the order of group is 4. This question is asked in [10] that whether the equivalence of the $\mathbb{Z}_4$-connectivity and the $\mathbb{Z}_2^2$-connectivity holds for 3-edge-connected graphs. The authors in this paper, together with Han in [5], gave a negative answer to this question as follows.

**Theorem 1.2** [5] *There exists a 3-edge-connected graph which is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected. Furthermore, there exists a 3-edge-connected graph which is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected.*

Both Theorem 1.1 and Theorem 1.2 are proved with involvements of computers, which are based on the computer-checked examples found in [6]. Hušek, Mohelníková and Šámal [6] asked whether one can find computer-free proofs for those results.

**Problem 1.3** [6] *Could we use a computer-free approach to prove that $\mathbb{Z}_4$-connectivity does not imply $\mathbb{Z}_2^2$-connectivity and neither vice versa?*

Langhede and Thomassen [13] studied the group connectivity and group coloring of planar graphs, and they were able to prove that $\mathbb{Z}_2^2$-connectivity does not imply $\mathbb{Z}_4$-connectivity that does not use computers.

**Theorem 1.4** [13] *There are infinitely many 3-edge-connected planar graphs which are $\mathbb{Z}_2^2$-connected, but not $\mathbb{Z}_4$-connected. Moreover, this result is verified by hand.*

In fact, Langhede and Thomassen [13] also proved that there are infinitely many 3-edge-connected planar graphs which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected. However, that proof is still based on the computer-checked examples found in [6] and uses another property about group coloring criticality that needs to be checked by computers.

In this paper, without any involvement of computers, we prove that $\mathbb{Z}_4$-connectivity does not imply $\mathbb{Z}_2^2$-connectivity.

**Theorem 1.5** *There are infinitely many 3-edge-connected graphs which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected. Furthermore, we are able to verify this result by hand.*

Theorem 1.5 is mainly based on some special graphs we constructed in Figure 4, and its proof applies some reductions on certain properties of the 4-cycles and the so-called collapsible graphs in the study of spanning Eulerian subgraph problem.

Note that Theorems 1.4 and 1.5 together answers Problem 1.3 completely. These results together provide also alternative proofs to Theorems 1.1 and 1.2 without any involvement of computers.

Given that $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity are distinct concepts, one might wonder what is the smallest graph whose $\mathbb{Z}_4$-connectivity differs from $\mathbb{Z}_2^2$-connectivity. The graphs constructed in [6] are small, and the minimal one has 11 vertices and 15 edges. However, we found the unique smallest graph whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity in this paper has 10 vertices and 14 edges, denoted $Q$ (see Figure 1). We prove the minimality and uniqueness through theoretical deductions and computer-assisted proofs.

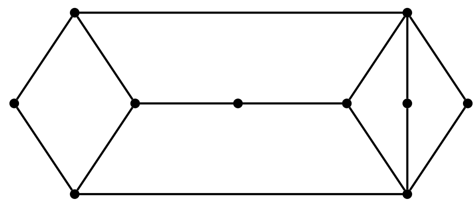


Figure 1: $Q$: The smallest $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected graph

**Theorem 1.6** *The graph $Q$ is the unique smallest graph whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity.*

The organization of the rest of this paper is as follows. We provide some preliminaries in section 2, and then we present the computer-free proof of Theorem 1.5 and the computer-aided proof of Theorem 1.6 in sections 3 and 4, respectively. Finally, we conclude this paper with a few remarks in section 5.

# 2 Preliminaries

## 2.1 Properties of group connectivity

The following lemma, due to Jaeger et al. [7], provides an equivalent definition of $\Gamma$-connectivity.

**Lemma 2.1** [7] *Let $G$ be a graph and $D$ an orientation of $G$. The following statements are equivalent:*

*(1) $G$ is $\Gamma$-connected.*

*(2) For any mapping $\bar{f}$ from $E(G)$ to $\Gamma$, there is a $\Gamma$-flow $(D, f)$ of $G$ such that $f(e) \neq \bar{f}(e)$ for each $e \in E(G)$.*

A cycle with $k$ vertices is referred to a *$k$-cycle*, written as $C_k$. Denote $G/H$ the graph obtained from $G$ by contracting edges in $H$. We refer readers to [10] for a survey about group connectivity. Some basic properties related to $\Gamma$-connected graphs are shown below.

**Lemma 2.2** [9] *Let $G$ be a graph and let $H$ be a subgraph of $G$.*

*(1) If both $H$ and $G/H$ are $\Gamma$-connected, then $G$ is $\Gamma$-connected.*

*(2) If $G$ is $\Gamma$-connected and $e \in G$, then $G/e$ is $\Gamma$-connected.*

*(3) The $k$-cycle $C_k$ is $\Gamma$-connected for any Abelian group $\Gamma$ with size $|\Gamma| \geq k+1$.*

Let $G_1$ and $G_2$ be two graphs with $u_1v_1 \in E(G_1)$ and $u_2, v_2 \in V(G_2)$. The graph obtained from $G_1$ and $G_2$ by deleting $u_1v_1$, identifying $u_1$ with $u_2$ as a new vertex $u$, and identifying $v_1$ with $v_2$ as a new vertex $v$ is called the 2-*sum* of $G_1$ and $G_2$ on $u_1v_1$ and $u_2, v_2$, denoted $G_1(u_1v_1) \bigoplus G_2(u_2, v_2)$ and written simply as $G_1 \bigoplus G_2$.

**Lemma 2.3** [5] *Let $G_1$ and $G_2$ be two graphs which are not $\Gamma$-connected. Then $G_1 \bigoplus G_2$ is not $\Gamma$-connected.*

## 2.2 Group connectivity of some special graphs

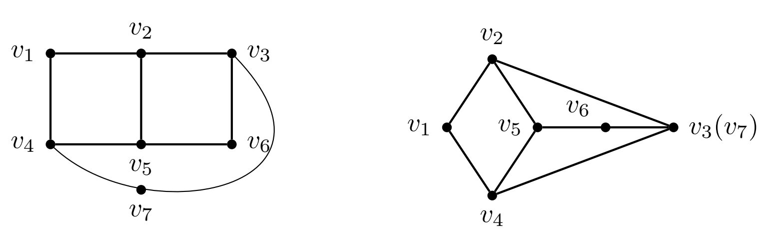
We introduce $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity for some special graphs in this subsection, which will be used later.

The degree of a vertex $v$ in the graph $G$ is denoted by $d_G(v)$. A graph is *trivial* if it has just one vertex, otherwise it is called *nontrivial.* A graph $G$ is *collapsible* if for any subset $S$ of $V(G)$ with $|S|$ even, there is a spanning connected subgraph $H_S$ of $G$ such that for any vertex $v \in V(H_S)$, $d_{H_S}(v)$ is odd if and only if $v \in S$. A relationship between collapsible graphs and $\Gamma$-connectivity is shown in the following lemma, due to Lai [8] in 1999.

**Lemma 2.4** [8] *Every collapsible graph is both $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected.*

The concept of collapsible graphs was first introduced by Catlin [2] in 1988 as a reduction method to find spanning Eulerian subgraphs. For more properties of collapsible graphs, we refer to [3, 12].

**Lemma 2.5** [4] *The graph shown in Figure 2a, denoted $F_2$, is collapsible, thus $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected.*




(a) $F_2$ (b) $F_3 \cong F_2/v_3v_7$

Figure 2: The graph $F_2$ and the graph $F_2/v_3v_7$

In fact, Lemma 2.5 can be proved by using the so-called $\pi$-reduction applied on 4-cycles in the reduction theory of collapsible graphs. Similarly, using $\pi$-reduction, we can also verify that the graph $P/e$, obtained from the Petersen graph $P$ by contracting an arbitrary edge $e$, is collapsible, and thus both $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected (see [4, 12]). On the other hand, it is well-known that the Petersen graph has no nowhere-zero 4-flow, since Tutte [17] proved that a cubic graph $G$ has a nowhere-zero 4-flow if and only if $G$ is 3-edge-colorable. Thus we can deduce the following lemma, which is also contained in a special case of Lemma 4.4 in the next section.

**Lemma 2.6** *The Petersen graph $P$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected, whereas $P/e$ is both $\mathbb{Z}_4$-connected and $\mathbb{Z}_2^2$-connected for any $e \in E(P)$.*

## 3 Proof of Theorem 1.5 avoiding computers

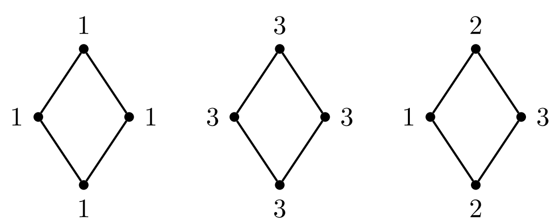
In this section, we present a $\mathbb{Z}_4$-connected, but non-$\mathbb{Z}_2^2$-connected planar graph avoiding computer-aids, shown in Figure 4, named $J_1$. In addition, we extended $J_1$ to a family with infinitely many graphs, see Definition 3.5. Furthermore, we construct a 3-edge-connected graph $M$ which is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected. The related proofs in this section are all computer-free.

We use $E_G(v)$ to denote the set of edges incident to vertex $v$ in graph $G$ and refer to a vertex of degree $k$ as a *$k$-vertex*. Firstly, we introduce several observations as follows.

**Observation 3.1** *Let $G$ be a $\mathbb{Z}_2^2$-connected graph and $(D, f)$ be a $(\mathbb{Z}_2^2, \beta)$-flow of $G$. The following statements hold.*

*(1) For a 3-vertex $v \in V(G)$ with $E_G(v) = \{e_1, e_2, e_3\}$, if $\beta(v) = (0, 0)$, then $f(e_1)$, $f(e_2)$, $f(e_3)$ are pairwise distinct, i.e. $\{f(e_1), f(e_2), f(e_3)\} = \{(0, 1), (1, 0), (1, 1)\}$.*

*(2) If $v$ is a 2-vertex adjacent to $u$ and $w$ in $G$, then $f(vu) \neq \beta(v)$ and $f(vw) \neq \beta(v)$.*




(a) $\beta_1$ (b) $\beta_2$ (c) $\beta_3$

Figure 3: The bad boundaries of $C_4$

Let $v_1, v_2, ..., v_k$ be the vertices of $C_k$ in a cyclic order. We use $[a_1, a_2, ..., a_k]$ to denote the mapping $v_i \mapsto a_i$, where $i \in \{1, 2, ..., k\}$. Denote by $\langle\beta\rangle = \langle\beta(v_1), \beta(v_2), ..., \beta(v_k)\rangle$ the set of

| $\beta$ | $\alpha$ | $\beta+\alpha$ |
| --- | --- | --- |
| $[1,1,1,1]$ | $[0,2,0,2]$ | $[1,3,1,3]$ |
| $[1,1,1,1]$ | $[0,3,2,3]$ | $[1,0,3,0]$ |
| $[3,3,3,3]$ | $[0,1,2,1]$ | $[3,0,1,0]$ |
| $[3,3,3,3]$ | $[0,2,0,2]$ | $[3,1,3,1]$ |
| $[1,2,3,2]$ | $[0,2,0,2]$ | $[1,0,3,0]$ |
| $[1,2,3,2]$ | $[0,1,2,1]$ | $[1,3,1,3]$ |
| $[2,3,2,1]$ | $[0,3,2,3]$ | $[2,2,0,0]$ |
| $[2,3,2,1]$ | $[0,1,2,1]$ | $[2,0,0,2]$ |
| $[3,2,1,2$ | $[0,3,2,3]$ | $[3,1,3,1]$ |
| $[3,2,1,2]$ | $[0,2,0,2]$ | $[3,0,1,0]$ |
| $[2,1,2,3]$ | $[0,3,2,3]$ | $[2,0,0,2]$ |
| $[2,1,2,3]$ | $[0,1,2,1]$ | $[2,2,0,0]$ |

Table 1: Details for Observation 3.3 (3)-(5)

all cyclic sequences of $[\beta(v_1), \beta(v_2), ..., \beta(v_k)]$; for example, $\langle 1,2,3,4\rangle = \{[1,2,3,4],[2,3,4,1],[3,4,1,2],[4,1,2,3]\}$. Denote by $\langle\beta_1\rangle + \langle\beta_2\rangle$ the set of all $\beta_1' + \beta_2'$ for every $\beta_1' \in \langle\beta_1\rangle$ and $\beta_2' \in \langle\beta_2\rangle$. We say $\langle\beta\rangle \in Z(C_k, \Gamma)$ if $\beta \in Z(C_k, \Gamma)$. Obviously, if $\beta \in Z(C_k, \Gamma)$, then every mapping in $\langle\beta\rangle$ is in $Z(C_k, \Gamma)$. We have following observations.

**Observation 3.2** *Define $\beta_1, \beta_2, \langle\beta_3\rangle \in Z(C_4, \mathbb{Z}_4)$ as: $\beta_1 = [1,1,1,1]$, $\beta_2 = [3,3,3,3]$, $\langle\beta_3\rangle = \langle 1,2,3,2\rangle$, see Figure 3. Then $C_4$ has a $(\mathbb{Z}_4, \beta)$-flow if and only if $\beta \notin \{\beta_1, \beta_2\} \cup \langle\beta_3\rangle$. Each mapping in $\{\beta_1, \beta_2\} \cup \langle\beta_3\rangle$ is referred to as a '* bad boundary *' of $C_4$.*

**Observation 3.3** *Let $\beta$ be a bad boundary of $C_4$. Then each of the following holds.*

*(1) Any mapping in $\langle\beta\rangle + \langle 2,2,0,0\rangle$ is not a bad boundary of $C_4$.*

*(2) Any mapping in $\langle\beta\rangle + \langle 1,0,3,0\rangle$ is not a bad boundary of $C_4$.*

*(3) At least one of the two sets $\langle\beta\rangle + [0,1,2,1]$ and $\langle\beta\rangle + [0,2,0,2]$ does not produce bad boundaries of $C_4$.*

*(4) At least one of the two sets $\langle\beta\rangle + [0,1,2,1]$ and $\langle\beta\rangle + [0,3,2,3]$ does not produce bad boundaries of $C_4$.*

*(5) At least one of the two sets $\langle\beta\rangle + [0,2,0,2]$ and $\langle\beta\rangle + [0,3,2,3]$ does not produce bad boundaries of $C_4$.*

We refer to Table 1 for checking the details of Observation 3.3 (3)-(5). For example, when

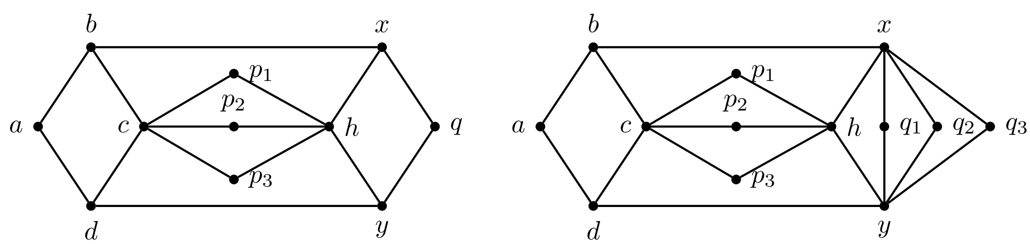
$\langle\beta\rangle = \langle 1,2,3,2\rangle$ we have $\langle\beta\rangle + [0,1,2,1] = \{[1,3,1,3],[2,0,0,2],[3,0,1,0],[2,2,0,0]\}$ and $\langle\beta\rangle + [0,2,0,2] = \{[1,0,3,0],[3,2,1,2]\}$, and so the set $\langle\beta\rangle + [0,1,2,1]$ contains no bad boundary. Furthermore, we can also check this fact similarly when $\beta = [1,1,1,1]$ or $[3,3,3,3]$, and hence Observation 3.3(3) holds in this case.

Let $D$ be a digraph. The set of arcs of $D$ is written as $A(D)$, and we say $u$ *dominates* $v$ if $uv \in A(D)$.

**Observation 3.4** *Let $G \cong K_{2,t}$, where $u, v$ are the $t$-vertices of $G$ and $x_1, x_2, ..., x_t$ are the 2-vertices of $G$. Suppose $D$ is an orientation of $G$, where $u$ dominates $x_i$, and $x_i$ dominates $v$ for any integer $i \in \{1,...,t\}$. If $t \geq 3$, then for any mapping $f : E(G) \to \mathbb{Z}_4 - \{0\}$, there is a mapping $f' : E(G) \to \mathbb{Z}_4 - \{0\}$ such that*

$$\partial f'(w) = \begin{cases} \partial f(w) + 2, & \text{if } w = u, v; \\ \partial f(w), & \text{if } w \in V(G) - \{w, v\}. \end{cases} \tag{1}$$

**Proof.** If there exists an integer $i \in \{1,...,t\}$ such that $f(ux_i) \neq 2$ and $f(x_i v) \neq 2$, then the observation follows from defining $f'(ux_i) = f(ux_i) + 2$, $f'(x_i v) = f(x_i v) + 2$, and $f'(e) = f(e)$ for any other edge $e$. Otherwise, it follows that $f(ux_i) = 2$ or $f(x_i v) = 2$ for every $i \in \{1,2,...,t\}$. Then there exists $y_i \in \{1,3\}$ such that $f(ux_i) + y_i \neq 0$ and $f(x_i v) + y_i \neq 0$ for any $i \in \{1,2,...,t\}$. Hence, there exists $y'_j \in \{0,1,3\}$ for $j = 1,2,3$ satisfying $f(ux_j) + y'_j \neq 0$, $f(x_j v) + y'_j \neq 0$ and $y'_1 + y'_2 + y'_3 = 2$. Let $f'(ux_j) = f(ux_j) + y'_j$, $f'(x_j v) = f(x_j v) + y'_j$, where $1 \leq j \leq 3$, and $f'(e) = f(e)$ for others. Then (1) follows. ■




(a) $J_1$ (b) $J_2$

Figure 4: The graph $J_1$ and $J_2$

**Definition 3.5** *We construct a graph family $\mathcal{J}$ as follows.*

*(1) $J_1 \in \mathcal{J}$ (see Figure 4b).*

*(2) For a graph $G \in \mathcal{J}$, if $G'$ is a graph obtained from $G$ by adding 2-vertices adjacent to $c$ and $h$; or adjacent to $x$ and $y$ (see Figure 5), then $G' \in \mathcal{J}$.*

**Theorem 3.6** *Each graph in $\mathcal{J}$ is $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected.*

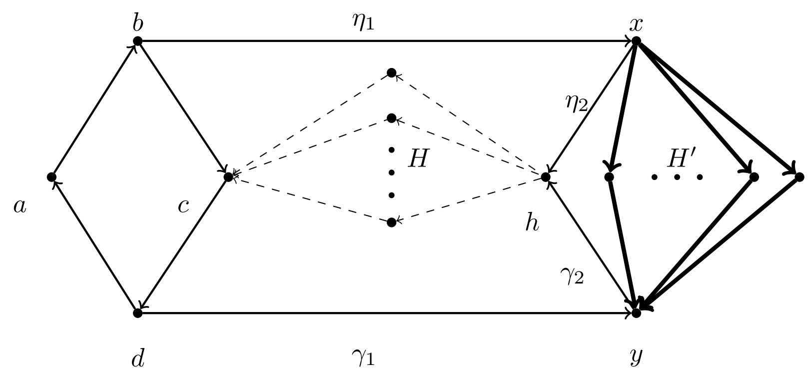



Figure 5: An oriented graph in $\mathcal{J}$, where $H$ and $H'$ denote subgraphs induced by dashed lines and thick solid lines, respectively

We would prove Theorem 3.6 by the following two lemmas.

**Lemma 3.7** *Any graph in $\mathcal{J}$ is not $\mathbb{Z}_2^2$-connected.*

**Proof.** We first claim that $J_1$ is not $\mathbb{Z}_2^2$-connected. Suppose, by contradiction, that $J_1$ is $\mathbb{Z}_2^2$-connected. Define $\beta \in Z(J_1, \mathbb{Z}_2^2)$ as:

$$\beta(v) = \begin{cases} (0,0), & \text{if } v = b, x, h; \\ (0,1), & \text{if } v = y, q; \\ (1,0), & \text{if } v = a, d; \\ (1,1), & \text{if } v = c, p_1, p_2, p_3. \end{cases}$$

Then there is a $(\mathbb{Z}_2^2, \beta)$-flow of $J_1$, written as $(D, f)$. Because the additive inverse of every element in $\mathbb{Z}_2^2$ is itself, it follows that $\partial f(v) = \sum_{uv \in E(G)} f(uv) = \beta(v)$ for each $v \in V(G)$. Thus, we may omit the orientation $D$. Since $p_i$ is a 2-vertex and $\beta(p_i) = (1,1)$, by Observation 3.1, we have $f(cp_i) \in \{(0,1),(1,0)\}$ for $i = 1,2,3$. Therefore, $\sum_{i=1,2,3} f(cp_i) \in \{(0,1),(1,0)\}$, and thus $f(bc) + f(cd) = \beta(c) - \sum_{i=1,2,3} f(cp_i) \in \{(0,1),(1,0)\}$. By Observation 3.1, we also have $f(ab) \in \{(0,1),(1,1)\}$. We shall discuss this two cases.

(I). If $f(ab) = (1,1)$, then $f(ad) = (0,1)$. Furthermore, this implies $f(bc), f(cd) \in \{(0,1),(1,0)\}$ by considering $\beta(b)$ and $\beta(d)$. But this contradicts $f(bc)+f(cd) \in \{(0,1),(1,0)\}$. Hence, $f(ab) \neq (1,1)$.

(II). If $f(ab) = (0,1)$, then $f(ad) = (1,1)$. By Observation 3.1, we have $f(bc) \in \{(1,1),(1,0)\}$, $f(cd) \in \{(1,1),(1,0)\}$, $f(xq) \neq (0,1)$ and $f(bx) \neq (0,1)$. So we must have

$f(bc)+f(cd)=(0,1)$, $\sum_{i=1,2,3} f(cp_i)=(1,0)$ and $f(xh)=(0,1)$. Since $\sum_{i=1,2,3} hp_i = \sum_{i=1,2,3} cp_i+(1,1)+(1,1)+(1,1)=(0,1)$, we have $f(hy)=\beta(h)-f(xh)-\sum_{i=1,2,3} hp_i = (0,0)$, a contradiction.

In conclusion, $f(ab)\notin\{(0,1),(1,1)\}$, a contradiction. So $J_1$ is not $\mathbb{Z}_2^2$-connected.

Let $G(k_1,k_2)$ be an arbitrary graph in $\mathcal{J}$ with $k_1$ 2-vertices adjacent to $c$ and $h$, named $p_i(1\le i\le k_1)$; and $k_2$ 2-vertices adjacent to $x$ and $y$, named $q_j(1\le j\le k_2)$, where $k_1\ge 3$ and $k_2\ge 1$. Now we are going to prove that $G(k_1,k_2)$ is not $\mathbb{Z}_2^2$-connected by induction on $k_1+k_2$.

Define $\beta'\in Z(G(k_1,k_2),\mathbb{Z}_2^2)$ as:

$$\beta'(v)=\begin{cases}(1,1), & \text{if } v=p_i \text{ for } i\in\{1,2,...,k_1\};\\ (0,1), & \text{if } v=q_j \text{ for } j\in\{1,2,...,k_2\};\\ \sum_{4\le s\le k_1}(0,1), & \text{if } v=h;\\ \sum_{2\le s\le k_2}(1,1), & \text{if } v=x;\\ (0,1)+\sum_{2\le s\le k_2}(1,0), & \text{if } v=y;\\ (1,1)+\sum_{4\le s\le k_1}(1,0), & \text{if } v=c;\\ \beta(v), & \text{otherwise.}\end{cases}$$

First of all, $G(3,1)=J_1$ has no $(\mathbb{Z}_2^2,\beta')$-flow by the argument above. By method of induction, we assume $G(k_1,k_2)$ has no $(\mathbb{Z}_2^2,\beta')$-flow. Then we use this fact to show that $G(k_1+1,k_2)$ has no $(\mathbb{Z}_2^2,\beta')$-flow and $G(k_1,k_2+1)$ has no $(\mathbb{Z}_2^2,\beta')$-flow. Suppose, by contradiction, that $G(k_1+1,k_2)$ has a $(\mathbb{Z}_2^2,\beta')$-flow $f$. If $f(cp_i)=(1,0)$ for some $i$, then we have $f(p_ih)=(0,1)$, and by deleting $cp_i$ and $p_ih$, we obtain a $(\mathbb{Z}_2^2,\beta')$-flow of $G(k_1,k_2)$, a contradiction. Therefore, we get $f(cp_i)=(0,1)$ and $f(p_ih)=(1,0)$ for each $i$ with $1\le i\le k_1+1$. Then, the following mapping $f':E(G(k_1,k_2))\to\mathbb{Z}_2^2-\{0\}$ is a $(\mathbb{Z}_2^2,\beta')$-flow of $G(k_1,k_2)$, where $f'$ is defined as follows.

$$f'(e)=\begin{cases}f(e)+(1,1), & \text{if } e\in\{cp_1,\ p_1h\};\\ f(e), & \text{if } e\in E(G(k_1,k_2))-\{cp_1,\ p_1h\}.\end{cases}$$

This contradicts the assumption that $G(k_1,k_2)$ has no $(\mathbb{Z}_2^2,\beta')$-flow. Hence we have proved that $G(k_1+1,k_2)$ has no $(\mathbb{Z}_2^2,\beta')$-flow. Using the same argument, it will also lead to a contradiction that $G(k_1,k_2)$ has a $(\mathbb{Z}_2^2,\beta')$ flow if we assume that $G(k_1,k_2+1)$ has a $(\mathbb{Z}_2^2,\beta')$-flow. Thus for any $k_1\ge 3$ and $k_2\ge 1$, $G(k_1,k_2)$ is not $\mathbb{Z}_2^2$-connected, i.e., any graph in $\mathcal{J}$ is not $\mathbb{Z}_2^2$-connected. ■

**Lemma 3.8** *Every graph in $\mathcal{J}$ is $\mathbb{Z}_4$-connected.*

**Proof.** Suppose $G \in \mathcal{J}$ and $D$ is an orientation of $G$ as shown in Figure 5. Let $H \cong K_{2,t_1}$ ($t_1 \geq 3$) and $H' \cong K_{2,t_2}$ ($t_2 \geq 3$) denote subgraphs of $G$ indicated by dashed lines and thick solid lines, respectively, in Figure 5. Let $C_4$ be the subgraph of $G$ induced by $\{a, b, c, d\}$ and let $F_1 = G/C_4$. Denote by $w$ the vertex contracted by $C_4$ in $F_1$. We first claim that $F_1$ is $\mathbb{Z}_4$-connected.

**Claim 1** *The graph $F_1$ is $\mathbb{Z}_4$-connected.*

**Proof.** By Lemma 2.5, the graph $F_2$ (Figure 2a) is $\mathbb{Z}_4$-connected. Denote $F_3 = F_2/(v_3v_7)$, shown in Figure 2b. Then $F_3$ is $\mathbb{Z}_4$-connected by Lemma 2.2 (2). One may observe that $F_3$ is a subgraph of $F_1$ and $F_1/F_3$ is $\mathbb{Z}_4$-connected. By Lemma 2.2 (1), it follows that $F_1$ is also $\mathbb{Z}_4$-connected. ■

Let $\beta \in Z(G, \mathbb{Z}_4)$. Define $\beta' \in Z(F_1, \mathbb{Z}_4)$ as follows:

$$\beta'(v) = \begin{cases} \beta(a) + \beta(b) + \beta(c) + \beta(d), & \text{if } v = w; \\ \beta(v), & \text{if } v \in V(F_1) - \{w\}. \end{cases}$$

By Claim 1, $F_1$ is $\mathbb{Z}_4$-connected, and thus there is a $(\mathbb{Z}_4, \beta')$-flow $(D|_{F_1}, f_1)$ of $F_1$ such that $f_1 : E(F_1) \to \mathbb{Z}_4 - \{0\}$ and $\partial f_1 = \beta'$. Let $v_1, ..., v_{t_1}$ be 2-vertices of $H$. Let $\eta_1 = f_1(bx)$, $\eta_2 = f_1(xh)$, $\gamma_1 = f_1(dy)$, $\gamma_2 = f_1(hy)$ and $\xi = \sum_{1 \leq i \leq t_1} f_1(cv_i)$. Define $\beta'' \in Z(C_4, \mathbb{Z}_4)$ as:

$$\beta''(v) = \begin{cases} \beta(a), & \text{if } v = a; \\ \beta(b) - \eta_1, & \text{if } v = b; \\ \beta(c) + \xi, & \text{if } v = c; \\ \beta(d) - \gamma_1, & \text{if } v = d. \end{cases}$$

**Claim 2** *If $\beta''$ is not a bad boundary of $C_4$, then $G$ has a $(\mathbb{Z}_4, \beta)$-flow.*

**Proof.** If $\beta''$ is not a bad boundary of $C_4$, then there is a $(\mathbb{Z}_4, \beta'')$-flow $(D|_{C_4}, f_2)$ of $C_4$ by Observation 3.2, where $D|_{C_4}$ is the restriction of $D$ on $C_4$. Therefore, $(D, f_1 \cup f_2)$ is a $(\mathbb{Z}_4, \beta)$-flow of $G$ as required. ■

Thus we may assume that $\beta''$ is a bad boundary of $C_4$. We are going to apply Observation 3.4 to modify the values of $\eta_1, \eta_2, \gamma_1, \gamma_2, \xi$ to obtain a new $(\mathbb{Z}_4, \beta')$-flow of $F_1$ so that the new $\beta''$ on $C_4$ is not a bad boundary anymore.

**CASE A.** Assume that either $\eta_1, \eta_2 \neq 2$ or $\gamma_1, \gamma_2 \neq 2$. We first replace $\xi$ with $\xi + 2$ by way of Observation 3.4. Now we consider the following two cases. If $\eta_1, \eta_2 \neq 2$, then replace $\eta_1, \eta_2$ with $\eta_1 + 2, \eta_2 + 2$. Otherwise, $\gamma_1, \gamma_2 \neq 2$, then replace $\gamma_1, \gamma_2$ by $\gamma_1 + 2, \gamma_2 + 2$. In both cases, the new $f_1$ is still a $(\mathbb{Z}_4, \beta')$-flow of $F_1$, and by Observation 3.3 (1), any mapping in $\langle \beta'' \rangle + \langle 2, 2, 0, 0 \rangle$ is not a bad boundary of $C_4$, i.e., the new $\beta''$ is not a bad boundary of $C_4$ anymore. Hence, $G$ has a $(\mathbb{Z}_4, \beta'')$-flow by Claim 2.

**CASE B.** Assume that at least one of $\eta_1, \eta_2$ is 2, and at least one of $\gamma_1, \gamma_2$ is 2.

(b1). If $\{\eta_1, \eta_2\} \neq \{\gamma_1, \gamma_2\}$ or $\{\eta_1, \eta_2\} = \{\gamma_1, \gamma_2\} = \{2, 2\}$ , then replace $\eta_1, \eta_2, \gamma_1, \gamma_2$ with $\eta_1 + q_1, \eta_2 + q_1, \gamma_1 + q_2, \gamma_2 + q_2$, where $\{q_1, q_2\} = \{1, 3\}$ such that $q_1 + \eta_1$, $q_1 + \eta_2$, $q_2 + \gamma_1$, $q_2 + \gamma_2$ are nonzero. Then we have obtained a new $(\mathbb{Z}_4, \beta')$-flow of $F_1$. Moreover, the new boundary of $C_4$ is in $\langle \beta''(a), \beta''(b), \beta''(c), \beta''(d) \rangle + \langle 1, 0, 3, 0 \rangle$, thus is not a bad boundary of $C_4$ by Observation 3.3 (2).

(b2). Otherwise, we have $\{\eta_1, \eta_2\} = \{\gamma_1, \gamma_2\} \neq \{2, 2\}$, and we provide a list of the specific cases. Table 2 shows the corresponding changes on $\eta_1, \eta_2, \gamma_1, \gamma_2, H, H'$ and the boundary $[\beta''(a), \beta''(b), \beta''(c), \beta''(d)]$, where $\Delta(\eta_1, \eta_2, \gamma_1, \gamma_2)$ and $\Delta[\beta''(a), \beta''(b), \beta''(c), \beta''(d)]$ indicate the changes of $(\eta_1, \eta_2, \gamma_1, \gamma_2)$ and $[\beta''(a), \beta''(b), \beta''(c), \beta''(d)]$, and 'Yes' means we need to adjust the graph $H$ or $H'$ by Observation 3.4. For example, when $(\eta_1, \eta_2, \gamma_1, \gamma_2) = (1, 2, 1, 2)$, we replace $(\eta_1, \eta_2, \gamma_1, \gamma_2)$ with $(\eta_1, \eta_2, \gamma_1, \gamma_2) + \Delta(\eta_1, \eta_2, \gamma_1, \gamma_2) = (1, 2, 1, 2) + (1, 3, 1, 3) = (2, 1, 2, 1)$, and in the same time we adjust the graph $H'$ by Observation 3.4. Then the new boundary of $C_4$ is $[\beta''(a), \beta''(b), \beta''(c), \beta''(d)] + \Delta[\beta''(a), \beta''(b), \beta''(c), \beta''(d)] = [\beta''(a), \beta''(b), \beta''(c), \beta''(d)] + [0, 1, 0, 3]$, which is not a bad boundary of $C_4$ by Observation 3.3 (2). Similarly, when $(\eta_1, \eta_2, \gamma_1, \gamma_2) = (1, 2, 2, 1)$, we may either change $(\eta_1, \eta_2, \gamma_1, \gamma_2)$ to $(\eta_1, \eta_2, \gamma_1, \gamma_2) + (1, 1, 1, 1)$ or change it to $(\eta_1, \eta_2, \gamma_1, \gamma_2) + (2, 0, 0, 2)$, and along the same line we also modify $H$ or $H'$ by Observation 3.4 whenever needed. Then the boundary of $C_4$ is either $[\beta''(a), \beta''(b), \beta''(c), \beta''(d)] + [0, 1, 2, 1]$ or $[\beta''(a), \beta''(b), \beta''(c), \beta''(d)] + [2, 0, 0, 2]$, in which at least one of the two sets is not a bad boundary of $C_4$ by Observation 3.3 (3). Note that all the remaining cases are listed in Table 2. In the cases of $(\eta_1, \eta_2, \gamma_1, \gamma_2) \in \{(1, 2, 2, 1), (2, 1, 1, 2), (3, 2, 2, 3), (2, 3, 3, 2)\}$, we shall need two rounds of modification similarly as shown above.

According to Observation 3.3, the new $\beta''$ is not a bad boundary of $C_4$ anymore whatever the case is in Table 2. Moreover, $f_1$ is still a $(\mathbb{Z}_4, \beta')$-flow of $F_1$.

Combining Claim 2, we could get a $(\mathbb{Z}_4, \beta)$-flow of $G$ in both CASE A and CASE B. Hence, every graph $G$ in $\mathcal{J}$ is $\mathbb{Z}_4$-connected by the definition of $\mathbb{Z}_4$-connectivity . ■

Let $P_i$ denote the Petersen graph with a fixed edge $e_i$ for $i = 1, 2, 3, 4$. Denote by $M$ the graph obtained from the 2-sums of $J_2$(Figure 4) and $P_1$, $P_2$, $P_3$, $P_4$ on $p_1, p_2$; $p_2, p_3$; $a, q_1$;

| $(\eta_1,\eta_2,\gamma_1,\gamma_2)$ | $\Delta(\eta_1,\eta_2,\gamma_1,\gamma_2)$ | $\Delta[\beta''(a),\beta''(b),\beta''(c),\beta''(d)]$ | $H$ | $H'$ |
|---|---|---|---|---|
| (1,2,1,2) | (1,3,1,3) | [0,1,0,3] | No | Yes |
| (2,1,2,1) | (3,1,3,1) | [0,3,0,1] | No | Yes |
| (1,2,2,1) | (1,1,1,1) | [0,1,2,1] | Yes | No |
| (1,2,2,1) | (2,0,0,2) | [0,2,0,2] | No | Yes |
| (2,1,1,2) | (1,1,1,1) | [0,1,2,1] | Yes | No |
| (2,1,1,2) | (3,1,1,3) | [0,3,2,3] | Yes | Yes |
| (2,3,2,3) | (1,3,1,3) | [0,1,0,3] | No | Yes |
| (3,2,3,2) | (3,1,3,1) | [0,3,0,1] | No | Yes |
| (3,2,2,3) | (3,3,3,3) | [0,3,2,3] | Yes | No |
| (3,2,2,3) | (2,0,0,2) | [0,2,0,2] | No | Yes |
| (2,3,3,2) | (3,3,3,3) | [0,3,2,3] | Yes | No |
| (2,3,3,2) | (1,3,3,1) | [0,1,2,1] | Yes | Yes |

Table 2: CASE B (b2) $\{\eta_1,\eta_2\} = \{\gamma_1,\gamma_2\} \neq \{2,2\}$

$q_2, q_3$ and $e_1$, $e_2$, $e_3$, $e_4$, respectively. Clearly, by using Lemma 2.3, similar constructions could generate infinitely many such graphs.

**Theorem 3.9** *The graph $M$ is a 3-edge-connected $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected graph.*

**Proof.** The graph $M$ is obviously 3-edge-connected. By Lemma 2.6 and Theorem 3.6, neither the Petersen graph nor the graph $J_2$ is $\mathbb{Z}_2^2$-connected. So $M$ is not $\mathbb{Z}_2^2$-connected by Lemma 2.3. By Lemmas 2.2 and 2.6, $M/J_2$ is $\mathbb{Z}_4$-connected; combining Lemma 2.2 and Theorem 3.6, we conclude that $M$ is $\mathbb{Z}_4$-connected. Hence, $M$ is a 3-edge-connected $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected graph. ■

Note that all the proofs above are done by hand, including all the lemmas we used in the above sections.

# 4 The smallest $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected graph

In this section, we shall explore the smallest graph whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity. We will use both theoretical reductions and a search assisted by computers. Due to the huge number of graphs and the complexity of testing group connectivity, we shall apply a few lemmas below to screen out the majority of the graphs, which avoids the brute force search and makes the computation possible.

**Lemma 4.1** [15] *Let $\Gamma$ be an Abelian group with size $|\Gamma| = k \geq 4$ and let $G$ be a graph with $v, w \in V(G)$. Assume that $G'$ is a graph obtained from $G$ by adding a $(k-1)$-path, say $u_0u_1...u_{k-1}$, by identifying $u_0$ with $v$, and $u_{k-1}$ with $w$. Then $G$ is $\Gamma$-connected if and only if $G'$ is $\Gamma$-connected.*

**Lemma 4.2** [14] *Let $m \geq 3$ be an integer, and let $G$ be a graph. If $G$ is $\mathbb{Z}_m$-connected, then $(m-2)G$ has $m-1$ edge-disjoint spanning trees, where $(m-2)G$ is the graph obtained from $G$ by replacing each edge of $G$ with $m-2$ multi-edges. In particular, if $G$ is $\mathbb{Z}_4$-connected, then*

$$|E(G)| \geq \frac{3}{2}(|V(G)| - 1).$$

**Lemma 4.3** [15] *If $G$ is $\mathbb{Z}_2^2$-connected and $|V(G)| \geq 4$, then $|E(G)| \geq \frac{4}{3}|V(G)|$.*

**Lemma 4.4** [4] *Let $G$ be a connected graph containing no nontrivial collapsible subgraph. If $|V(G)| \leq 11$ and $|E(G)| \geq 2|V(G)| - 5$, then $G$ belongs to a well-characterized graph family $\mathcal{F}$.*

Some graphs in $\mathcal{F}$ are shown in Figure 6 and more details are shown in Definition 4.5 and Lemma 4.6. Let $m$ and $n$ be two positive integers. The complete graph with $n$ vertices is denoted by $K_n$. The complete bipartite graph $G[X, Y]$ with parts of size $|X| = m$ and $|Y| = n$ is written as $K_{m,n}$.

**Definition 4.5** *The Petersen graph is denoted by $P$. Let $s_1, s_2, s_3, m, l, t$ be natural numbers with $m, l \geq 1$ and $t \geq 2$. Let $K_{1,3}$ be the complete bipartite graph, with the 3-vertex named $a$ and three 1-vertices named $a_1, a_2, a_3$ respectively. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from $K_{1,3}$ by adding $s_i$ vertices adjacent to $a_i$ and $a_{i+1}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with $u, u'$ being the nonadjacent vertices of degree $t$. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex $u''$ that joins to $u'$ only. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex $u''$ that joins to a 2-vertex of $K_{2,t}$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying $u$ with $w$, and $w''$ with $u'$; let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying $w, w''$ with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w, w', w'')$ by identifying $w, w''$ with two 2-vertices in $K_{2,m+2}$, respectively.*

*Define $\mathcal{F} = \{K_1, K_2, K_{2,t}, K'_{2,t}, K''_{2,t}, K_{1,3}(s_1, s_2, s_3), S(m, l), J(m, l), J'(m, l), P\}$, where $t, s_1, s_2, s_3, m, m, l$ are natural numbers.*

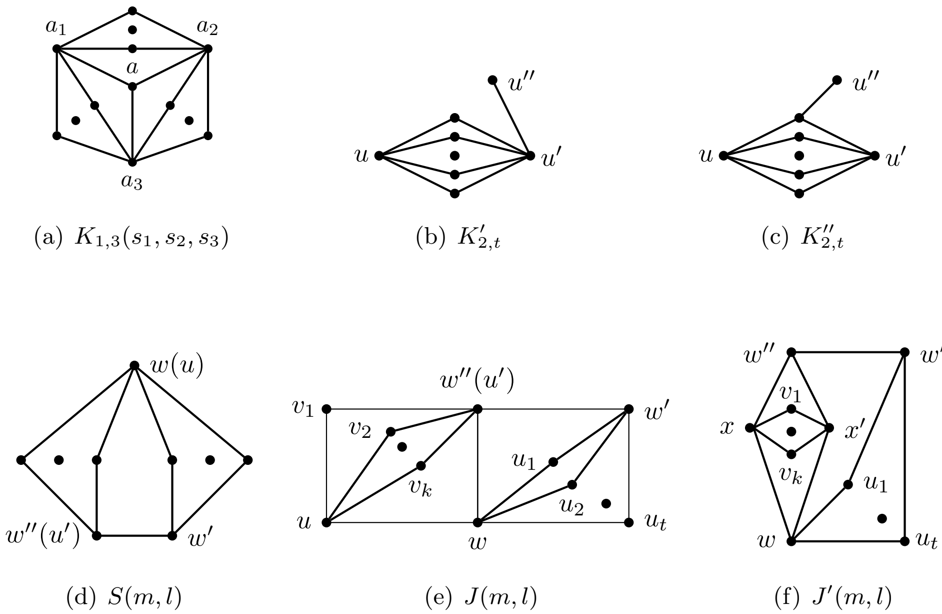



Figure 6: Part of the graphs in $\mathcal{F}$

We further present the properties of graphs in $\mathcal{F}$ in the following lemma, whose proof is shown in Appendices.

**Lemma 4.6** *Each nontrivial graph in $\mathcal{F}$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected.*

Recalling Theorem 1.6, we state it in another way as follows.

**Theorem 1.6′** *Let $G$ be a graph with $|V(G)| \leq 10$. If $G \cong Q$, then it is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected. If $G \ncong Q$, then $G$ is $\mathbb{Z}_4$-connected if and only if $G$ is $\mathbb{Z}_2^2$-connected.*

**Proof.** Suppose $G$ is one of the the smallest graph for which $\mathbb{Z}_4$-connectivity is not equivalent to $\mathbb{Z}_2^2$-connectivity on $|V(G)|$. We claim that $G$ has neither adjacent 2-vertices nor nontrivial collapsible subgraph.

**Claim 3** *The graph $G$ has no adjacent 2-vertices.*

**Proof.** Suppose, by way of contradiction, $G$ has a pair of adjacent 2-vertices, $u$ and $v$. Let $G' = G - \{u, v\}$. By Lemma 4.1, $G$ is $\Gamma$-connected if and only if $G'$ is $\Gamma$-connected for any Abelian group with $|\Gamma| = 4$. Obviously, $|V(G')| < |V(G)|$ contradicts the choice of $G$. Therefore, $G$ has no adjacent 2-vertices. ■

**Claim 4** *The graph $G$ contains no nontrivial collapsible subgraph.*

**Proof.** Without loss of generality, suppose $G$ is $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected (an analogous argument for $G$ is $\mathbb{Z}_2^2$-, but non-$\mathbb{Z}_4$-connected). By contradiction, assume $H$ is a nontrivial collapsible subgraph of $G$. Then $H$ is $\mathbb{Z}_4$-connected as well as $\mathbb{Z}_2^2$-connected by Lemma 2.4. Therefore, $G/H$ is $\mathbb{Z}_4$-connected. Because $G$ is the smallest graph for which the $\mathbb{Z}_4$-connectivity and $\mathbb{Z}_2^2$-connectivity is not equivalent, $G/H$ is $\mathbb{Z}_2^2$-connected. Hence, $G$ is $\mathbb{Z}_2^2$-connected by Lemma 2.2, a contradiction. ■

**Claim 5** *(1) $G$ contains no triangle.*
*(2) $G$ contains no vertex whose neighbors are all 2-vertices.*
*(3) Let $V_2$ be the set of all 2-vertices of $G$. Then $G - V_2$ is connected.*

**Proof.** As a triangle is collapsible, (1) follows from Claim 4. If $G$ contains a vertex $x$ whose neighbor set $N_G(x)$ consists of 2-vertices, then we contract $V(G)-X \cup N_G(x)$ in $G$ to obtain a complete bipartite graph $K_{2,t}$, where $t = |N_G(x)|$. Since $K_{2,t}$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected, we conclude by Lemma 2.2 that $G$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected, a contradiction. Hence (2) is true. For (3), it is similar to (2). Suppose that $G - V_2$ has at least two components, and let $H$ be a component of $G - V_2$. We contract $H$ into a vertex and contract $V(G) - V_2 - V(H)$ into another vertex, then we obtain a $K_{2,t}$, which implies that $G$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected, contradicting our assumption. Therefore, (3) holds. ■

Combining Claim 4, Lemmas 4.4 and 4.6, we have $|E(G)| < 2|V(G)| - 5$. Together with Lemmas 4.2 and 4.3, we have

$$\min\{\frac{4}{3}|V(G)|,\ \frac{3}{2}(|V(G)| - 1)\} \leq |E(G)| \leq 2|V(G)| - 6.$$

As $|V(G)| \leq 10$, the only two solutions of the inequality above are: (1) $|V(G)| = 9$ and $|E(G)| = 12$; (2) $|V(G)| = 10$ and $|E(G)| = 14$.

Denote by $n_k$ the number of $k$-vertices of $G$. By Claims 3 and 5, we have $n_2 \leq 4$ when $|V(G)| = 9$; $n_2 \leq 5$ when $|V(G)| = 10$. Now we are able to determine all possible degree sequences of $G$ as follows.

(1) $|V(G)| = 9$ and $|E(G)| = 12$.

(1.1) $n_2 = 4, n_3 = 4, n_4 = 1$;

(1.2) $n_2 = 3, n_3 = 6$.

(2) $|V(G)| = 10$ and $|E(G)| = 14$.

(2.1) $n_2 = 5, n_3 = 3, n_4 = 1, n_5 = 1$;

(2.2) $n_2 = 5, n_3 = 2, n_4 = 3$;

(2.3) $n_2 = 4, n_3 = 4, n_4 = 2$;

(2.4) $n_2 = 4, n_3 = 5, n_5 = 1$;

(2.5) $n_2 = 3, n_3 = 6, n_4 = 1$;

(2.6) $n_2 = 2, n_3 = 8$.

Table 3: The degree sequence and number of corresponding connected graphs

| Cases | Degree Sequence | Total | $\mathbb{Z}_4$&$\mathbb{Z}_2^2$ | non-$\mathbb{Z}_4$&non-$\mathbb{Z}_2^2$ | $\mathbb{Z}_4$&non-$\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$&non-$\mathbb{Z}_4$ |
|---|---|---|---|---|---|---|
| $\|V(G)\| = 9$ | (1.1) | 227 | 76 | 151 | 0 | 0 |
| $\|E(G)\| = 12$ | (1.2) | 63 | 21 | 42 | 0 | 0 |
| | (2.1) | 1625 | 896 | 729 | 0 | 0 |
| | (2.2) | 1183 | 652 | 531 | 0 | 0 |
| $\|V(G)\| = 10$ | (2.3) | 2187 | 1408 | 778 | **1** | 0 |
| $\|E(G)\| = 14$ | (2.4) | 664 | 404 | 260 | 0 | 0 |
| | (2.5) | 915 | 662 | 253 | 0 | 0 |
| | (2.6) | 113 | 88 | 25 | 0 | 0 |

Constructing all graphs corresponding to these degree sequences, we check the group connectivity of each graph by the pseudocode presented in Appendices. Then one may obtain Table 3, which shows the number of connected graphs with corresponding degree sequence. The unique $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected graph with degree sequence (2.3) : $n_2 = 4, n_3 = 4, n_4 = 2$ is shown in Figure 1. Thus, the graph $Q$ is the unique smallest graph whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity. ■

# 5 Conclusions and Remarks

In this paper, we found the unique smallest graph $Q$, whose $\mathbb{Z}_4$-connectivity varies from $\mathbb{Z}_2^2$-connectivity. Since $Q$ is a planar graph, the group connectivity of $Q$ and the group coloring of the dual graph of $Q$ (which has order 6) are equivalent. Let $D$ be an orientation of graph $G$. If for any mapping $\bar{\varphi} : E(G) \to \Gamma$, there is a vertex coloring $\varphi : V(G) \to \Gamma$ such that $\varphi(u) - \varphi(v) \neq \bar{\varphi}(uv)$ for any $uv \in A(D)$, then we say $G$ is $\Gamma$-*(group) colorable*. It is proved in [7] that a connected plane graph is $\Gamma$-colorable if and only if its dual graph is $\Gamma$-connected. Since $Q$ and $J_2$ are planar graphs, there are similar consequences of Theorems 1.6 and 3.6 on group coloring. However, both $Q$ and $J_2$ have 2-edge-cuts, thus their dual graphs are not simple. Langhede and Thomassen [13] studied the group connectivity of 3-edge-connected planar graphs, whose dual graphs are simple graphs. Other than Theorem 1.4, they also showed that there exists an infinite family of 3-edge-connected planar graphs, which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected, where its proof involves the help of computers. For more results on group connectivity of other groups, we refer the readers to [11, 13].

Theorems 3.6, 3.9, and 1.4 together provide a computer-free proof to Theorems 1.1 and 1.2, which solves the problem proposed by Hušek et al. [6]. However, the computer-free proof for the existence of 3-edge-connected planar graphs, which are $\mathbb{Z}_4$-connected, but not $\mathbb{Z}_2^2$-connected, remains open. Another open problem is to find the smallest $\mathbb{Z}_2^2$-connected and non-$\mathbb{Z}_4$-connected graph, which may or may not need the help of computer. Due to the absence of efficient reduction methods and the exponential growth of the number of graphs as their order increases, this problem is beyond our scope. Note that there is a $\mathbb{Z}_2^2$-connected and non-$\mathbb{Z}_4$-connected graph with 15 vertices and 21 edges found in [6].

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# Appendices

## A The proof of Lemma 4.6

**Lemma 4.6** *Each nontrivial graph in $\mathcal{F}$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected.*

**Proof.** Suppose $G \in \mathcal{F}$. If $G \cong P$, then it follows that $G$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected from Lemma 4.6. Since it is necessary to be 2-edge-connected for a graph admitting nowhere-zero 4-flows, $G$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected if $G \cong K_2, K'_{2,t}, K''_{2,t}$. If $G \cong S(m,l)$, then $G/w''w' \cong K_{2,t}$. So if $K_{2,t}$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected, then by Lemma 2.2, $G \cong S(m,l)$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected. Thus, it suffices to prove Lemma 4.6 for graph in $\{K_{2,t}, K_{1,3}(s_1,s_2,s_3), J(m,l), J'(m,l)\}$. We will first prove by way of contradiction that $G$ is not $\mathbb{Z}_4$-connected. Suppose $G$ is $\mathbb{Z}_4$-connected. Let $D$ be an orientation of $G$. By Lemma 2.1, for any $\bar{f}: E(G) \to \mathbb{Z}_4$, there is a $\mathbb{Z}_4$-flow $(D,f)$ of $G$ such that $f(e) \neq \bar{f}(e)$ for each $e \in E(G)$. We denoted by $[k_1,k_2]$ the set of integers $i$ with $k_1 \leq i \leq k_2$. The labels on graphs that we discussed in the succeeding proofs are indicated in Figure 6.

Assume $G \cong K_{2,t}$. Let $u$ and $u'$ be two distinct $t$-vertices of $G$ and $\{v_1, v_2, ..., v_t\}$ be the set of 2-vertices of $G$. Let $D$ be an orientation of $G$ such that $u$ dominates $v_i$ and $v_i$ dominates $u'$ for $i \in [1,t]$. Define $\bar{f}: E(G) \to \mathbb{Z}_4$ as: $\bar{f}(uv_1) = 2$, $\bar{f}(u'v_1) = 0$, $\bar{f}(uv_i) = 1$, $\bar{f}(uv_i) = 3$ for $i \in [2,t]$. Since $\partial f(v_i) = 0$ for $i \in [1,t]$ and $f(e) \neq \bar{f}(e)$ for $e \in E(G)$, we have $f(uv_1)$ is odd and $f(uv_i)$ is even for $i \in [2,t]$. Therefore, $\partial f(v) = \sum_{1 \leq i \leq t} f(uv_i)$ is odd which contradicts $\partial f(v) = 0$. Hence, $G \cong K_{2,t}$ is not $\mathbb{Z}_4$-connected.

Assume $G \cong K_{1,3}(s_1,s_2,s_3)$. Let $u_i, v_j, w_k$ be the 2-vertices adjacent to $a_1, a_2$; $a_2, a_3$; $a_1, a_3$, respectively, where integers $i \in [1,s_1]$, $j \in [1,s_2]$ and $k \in [1,s_3]$. Let $D$ be an orientation of $G$ such that $a_1$ dominates $u_i$; $u_i$ dominates $a_2$; $a_2$ dominates $v_j$; $v_j$ dominates $a_3$; $a_3$ dominates $w_k$; $w_k$ dominates $a_1$; $a$ dominates $a_1, a_2$ and $a_3$. Suppose

$$\bar{f}(e) = \begin{cases} 0, & e = aa_1, aa_2, aa_3; \\ 1, & e \in \{a_1u_i, a_2v_j, a_3w_k \mid i \in [1,s_1],\ j \in [1,s_2],\ k \in [1,s_3]\}; \\ 3, & e \in \{a_2u_i, a_3v_j, a_1w_k \mid i \in [1,s_1],\ j \in [1,s_2],\ k \in [1,s_3]\}. \end{cases}$$

Then for any 2-vertex $v \in V(G)$ incident with $e_v^1$ and $e_v^2$, one may obtain that $f(e_v^1) = f(e_v^2)$ is even. So $f(aa_1), f(aa_2), f(aa_3)$ is even by $\partial f(a_1) = \partial f(a_2) = \partial f(a_3) = 0$. Furthermore, $f(aa_1) = f(aa_2) = f(aa_3) = 2$ as $\bar{f}(aa_1) = \bar{f}(aa_2) = \bar{f}(aa_3) = 0$. But this contradicts $\partial f(a) = 0$. It follows that $G \cong K_{1,3}(s_1,s_2,s_3)$ is not $\mathbb{Z}_4$-connected.

Assume $G \cong J(m,l)$. Denote by $v_1, v_2, ..., v_m$ and $u_1, u_2, ..., u_t$ all the 2-vertices adjacent to $u, w''$ and $w, w'$, respectively. Let $D$ be an orientation of $G$ such that $u$ dominates $v_i$ and $w$; $v_i$ dominates

$w''$; $w$ dominates $u_j$ and $w''$; $w''$ and $u_j$ dominates $w'$, where $i \in [1,m]$ and $j \in [1,l]$. Let

$$\bar{f}(e) = \begin{cases} 1, & e \in \{v_i w'', u_j w' \mid 1 \le i \le m,\ 1 \le j \le l\}; \\ 3, & e \in \{uv_i, wu_j \mid 1 \le i \le m,\ 1 \le j \le l\}; \\ 0, & e \in \{uw, ww'', w''w'\}. \end{cases}$$

Then $f(uv_i) = f(v_i w'')$ and $f(wu_j) = f(u_j w')$ are even. By $\partial f(u) = \partial f(w') = \partial f(w'') = 0$, we also have $f(uw)$, $f(ww'')$ and $f(w''w')$ are even. Since $f(e) \ne \bar{f}(e)$ for each $e \in E(G)$, there is $f(uw) = f(ww'') = f(w''w') = 2$. Therefore, $2 = \partial f(u) - f(uw) = \partial f(w) + f(uw) - f(ww'') = 0$, a contradiction. Hence, $G \cong J(m,l)$ is not $\mathbb{Z}_4$-connected.

Assume $G \cong J'(m,l)$. Let $v_1, v_2, ..., v_m$ and $u_1, u_2, ..., u_l$ denote all the 2-vertices adjacent to $x, x'$ and $w, w'$, respectively. Let $D$ be the orientation of $G$ such that $x, x'$ and $w'$ dominates $w''$; $x$ dominates $v_i$; $v_i$ dominates $x'$; $w'$ dominates $u_j$; $x, x', u_j$ dominates $w$, for $i \in [1,m]$ and $j \in [1,l]$. Define $\bar{f}$ as:

$$\bar{f}(e) = \begin{cases} 0, & e \in \{xv_1, wu_1\}; \\ 1, & e \in \{xv_i, w'u_j, xw, xw'' \mid 2 \le i \le m,\ 2 \le j \le l\}; \\ 2, & e \in \{v_1 x', u_1 w'\}; \\ 3, & e \in \{v_i x', u_j w, w''w', x'w'', x'w \mid 2 \le i \le m,\ 2 \le j \le l\}. \end{cases}$$

Then $f(xv_1) = f(v_1 x')$ and $f(wu_1) = f(u_1 w')$ are odd, and $f(xv_i) = f(v_i x')$ and $f(wu_j) = f(u_j w')$ are even for $i \in [2,m]$ and $j \in [2,l]$. Thus, $f(w'w'')$ is odd, furthermore, $f(w'w'') = 1$ because $\bar{f}(w'w'') = 3$. Moreover,

$$\begin{cases} f(xw) + f(xw'') = f(x'w) + f(x'w''), \\ f(xw'') + f(x'w'') = 3, \\ f(xw) + f(x'w) = 1. \end{cases} \tag{2}$$

One may obtain that $f(xw'') + f(xw)$ is even by (2). Therefore, $\partial f(x) = f(xw'') + f(xw) + f(xv_1) + \sum_{2 \le i \le m} f(xv_i)$ is odd, contradicting $\partial f(x) = 0$. Thus, $G \cong J'(m,l)$ is not $\mathbb{Z}_4$-connected.

In conclusion, any nontrivial graph in $\mathcal{F}$ is not $\mathbb{Z}_4$-connected. The proof of non-$\mathbb{Z}_2^2$-connectivity on $G$ has a similar flavor, so we only present the key ingredient. By contradiction, suppose $G$ is $\mathbb{Z}_2^2$-connected. Let $D$ be an orientation of $G$ as above in the corresponding case. By Lemma 2.1, for any $\bar{f} : E(G) \to \mathbb{Z}_2^2$, there is a $\mathbb{Z}_2^2$-flow $(D, f)$ of $G$ such that $f(e) \ne \bar{f}(e)$ for each $e \in E(G)$. Refer to Figure 6 for symbols of vertices. And define $\bar{f}$ as follows:

(a). If $G \cong K_{2,t}$, then $\bar{f}(uv_1) = (1,1)$, $\bar{f}(v_1 u') = (0,0)$, $\bar{f}(uv_i) = (1,0)$ and $\bar{f}(v_i u') = (0,1)$, for each integer $i \in [1,t]$;

(b). If $G \cong K_{1,3}(s_1, s_2, s_3)$, then $\bar{f}(aa_1) = \bar{f}(aa_3) = (1,1)$, $\bar{f}(aa_2) = (0,0)$, $\bar{f}(a_1 u_i) = \bar{f}(a_2 v_j) = \bar{f}(a_3 w_k) = (1,0)$ and $\bar{f}(u_i a_2) = \bar{f}(v_j a_3) = \bar{f}(w_k a_1) = (0,1)$, where integers $i \in [1,s_1]$, $j \in [1,s_2]$ and $k \in [1,s_3]$;

(c). If $G \cong J(m,l)$, then $\bar{f}(uw) = \bar{f}(ww'') = \bar{f}(w''w') = (0,0)$, $\bar{f}(uv_i) = \bar{f}(wu_j) = (0,1)$ and $\bar{f}(v_iw'') = \bar{f}(u_jw') = (1,0)$, where integers $i \in [1,m]$ and $j \in [1,l]$;

(d). If $G \cong J'(m,l)$, then $\bar{f}(x'w'') = (0,0)$, $\bar{f}(w''w') = \bar{f}(xw'') = (1,1)$, $\bar{f}(xv_i) = \bar{f}(wx) = \bar{f}(wu_j) = (1,0)$, $\bar{f}(v_ix') = \bar{f}(wx') = \bar{f}(u_jw') = (0,1)$, for $i \in [1,m]$ and $j \in [1,l]$.

One may check that there is no $\mathbb{Z}_2^2$-flow $(D,f)$ of $G$ such that $f(e) \neq \bar{f}(e)$ for each $e \in E(G)$, for any $G \in \{K_{2,l}, K_{1,3}(s_1,s_2,s_3), J(m,l), J'(m,l)\}$, which derives contradictions. Thus, as asserted, each nontrivial graph in $\mathcal{F}$ is neither $\mathbb{Z}_4$-connected nor $\mathbb{Z}_2^2$-connected. ■

# B Program used in Theorem 1.6

The implementation for checking whether a graph is $\mathbb{Z}_4$-connected or $\mathbb{Z}_2^2$-connected is referring to [6] R. Hušek, et al. (https://gitlab.kam.mff.cuni.cz/radek/group-connectivity-pub). And the pseudo-code[1] of checking the group connectivity of all graphs with given degree sequences is as follows, where f(G) is designed to check if $G$ is $\mathbb{Z}_4$-connected and non-$\mathbb{Z}_2^2$-connected.

**Algorithm 1:** Programm for Theorem 1.6

```
from groupConnectivity import *
from sage.graphs.connectivity import edge_connectivity
def f(G)
for v in G.vertex_iterator() do
    if not testGroupConnectivity(G, "Z2_2", useSubgraphs = False) and
     testGroupConnectivity(G, "Z4", useSubgraphs = False) then
        return True
    else
        return False
for seq in Table 3 do
    print [g.graph6_string() for g in graphs(|V(G)|, degree_sequence = seq) if
     g.is_connected() and f(g)]
```

[1] The pseudo-code is based on Python with Sage libraries.