Proofs of four generating function conjectures for arbor polytopes

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Abstract

This paper proves four conjectured generating series, due to Chapoton, which concern invariants of posets and polytopes associated with a specific sequence of arbors. Two of these conjectures provide closed-form formulas for the generating series of the Zeta polynomial and the generating series of the M-triangle of the poset, respectively. The remaining two conjectures pertain, respectively, to the Ehrhart polynomial and the Laplace transform of the volume function of the associated arbor polytope.

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1 Introduction

The combinatorial framework of *arbors* was recently introduced to link rooted trees with geometric and algebraic structures. Arbor polytopes were first introduced and studied by Chapoton [4]. Subsequently, the combinatorial properties of two special classes of arbor polytopes (i.e., the composition polytopes [1] and the polytopes $Q_{n,k}$ [2]) were investigated.

An *arbor* t is defined as a rooted tree where each vertex v is decorated by a non-empty subset of a finite set $[n] := \{1, 2, \ldots, n\}$, and these subsets form a partition of $[n]$. We call n the *size* of the arbor. The cardinality of a vertex v is denoted by $|v|$. For example, Figure 1 is an arbor on the set [8].

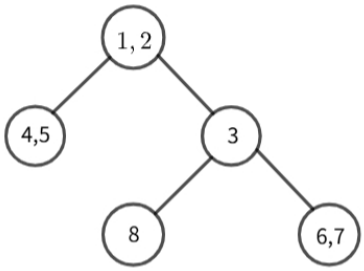


Figure 1: An arbor on the set [8].

To every arbor t on a set $[n]$, we associate a lattice polytope Q_t and a partial order P_t.

Definition 1.1 (The Polytope Q_t)**.** *The polytope Q_t is defined in the vector space $\mathbb{R}^n$ with coordinates $(x_i)_{i\in[n]}$ by the inequalities $x_i \geq 0$ for all $i \in [n]$ and*

$$\sum_{i\in\mathcal{D}(v)} x_i \leq |\mathcal{D}(v)|$$

for every vertex v, where $\mathcal{D}(v)$ is the set of labels in the sub-tree rooted at v.

For the example in Figure 1, besides all $x_i \geq 0$, the defining inequalities are

$$x_8 \leq 1, \quad x_6 + x_7 \leq 2, \quad x_3 + x_6 + x_7 + x_8 \leq 4, \quad x_4 + x_5 \leq 2, \quad x_1 + x_2 + \cdots + x_8 \leq 8.$$

A convex polytope $\mathcal{P}$ is called a *lattice polytope* if all of its vertices have integer coordinates. Chapoton [4, Lemma 1.2] proved that the arbor polytope is a simple lattice polytope.

Definition 1.2 (The Poset P_t)**.** *The poset P_t consists of the lattice points of Q_t ordered by coordinate-wise comparison: $x \leq y$ if $x_i \leq y_i$ for all $i \in [n]$. This poset is graded by the height function* $\mathrm{ht}(x) = \sum_{i\in[n]} x_i$.

The reader is referred to reference [9, Section 3] for the necessary knowledge on posets. A standard reference on the polytope is [3]. We are concerned with several classical invariants:

- **Ehrhart Polynomial** $E_t(u)$: Counts the number of lattice points in the dilated polytope uQ_t.
- **Zeta Polynomial** $Z_t(u)$: Counts chains of length $u-2$ in the poset P_t.
- M**-triangle** $M_t(X, Y)$: A polynomial defined using Möbius numbers and the rank function.
- **Laplace Transform of Volume** $\mathcal{L}_t(v)$: Encodes the volume of slices of Q_t according to height.

In this work, we focus on the specific arbor sequence t_n ($n \geq 1$), which consists of a root vertex with one element, supporting $n-1$ child vertices, each a leaf with one element. Figure 2 is the example of t_5. This sequence is closely related to Hochschild polytopes [5, 8]. The arbor polytope Q_{t_n} is also a special case of a class of polytopes studied in the literature [2].

Throughout this paper, $\mathbb{N}$ denotes the set of non-negative integers, and $\mathbb{Z}_+$ denotes the set of positive integers.

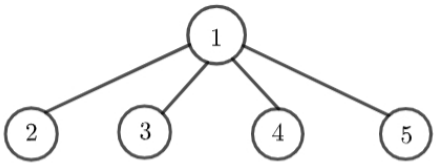


Figure 2: The example of t_5.

Using `FriCAS` [7], Chapoton [4, Section 9] conjectured four formulas for various generating series (which are formal power series) in s for the arbors t_n. The four formulas, respectively, involve

- the Zeta polynomials of the posets P_{t_n} (see Theorem 2.2),
- the M-triangles of the posets P_{t_n} (see Theorem 2.6),
- the Ehrhart polynomials of the polytopes Q_{t_n} (see Theorem 3.1),
- the Laplace transform of volume functions of the polytopes Q_{t_n} (see Theorem 3.3).

In this paper, we mainly prove the four conjectures proposed by Chapoton, namely, the four theorems mentioned above.

The paper is organized as follows. Section 2 introduces basic definitions on the Zeta polynomial and M-triangle of a poset. We then establish Theorems 2.2 and 2.6. In Section 3, we introduce the Ehrhart polynomial and the volume function of a lattice polytope. We then prove Theorems 3.1 and 3.3.

2 Generating Functions for Poset Invariants

2.1 Zeta Polynomials

Let P be a finite poset. If $u \geq 2$, then define $Z_P(u)$ to be the number of chains $e_1 \leq e_2 \leq \cdots \leq e_{u-1}$ in P. We call $Z_P(u)$ (regarded as a function of u) the *Zeta polynomial* of P. For the relevant properties of the Zeta polynomial, we refer the reader to [9, Section 3.12].

For an arbor t, Chapoton extends the Zeta polynomial of the poset P_t. Let $Z_t(u, X)$ be the polynomial in u and X such that for every integer $m \geq 2$,

$$Z_t(m, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_{m-1}} X^{\mathrm{ht}(e_{m-1})},$$

where ht is the height of an element. Setting $X = 1$ yields the classical Zeta polynomial of the poset P_t.

Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. The *generalized binomial coefficient* $\binom{\alpha}{k}$ is defined as:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

In particular, when $k = 0$, it is defined as 1. We will frequently use the fact that $\binom{n}{k} = 0$ for nonnegative integers n, k with $n < k$.

Let t be an arbor on a set of cardinality $n \geq 2$. Let $(t^{(1)}, \ldots, t^{(w)})$ be the sub-trees of the root vertex of t. Let r be the size of the root vertex of t. Chapoton [4] derived the following proposition.

Proposition 2.1. [4, Proposition 2.7] *Let* $Z_W = \prod_{k=1}^{w} Z_{t^{(k)}}(u, X)$. *Write* $Z_W = \sum_j W_j X^j$. *Then for* $0 \leq j \leq n$, *the coefficient of* X^j *in* $Z_t(u, X)$ *is*

$$\sum_{\ell=\max(0,j-n+r)}^{j} \binom{r(u-1)+\ell-1}{\ell} W_{j-\ell}.$$

The first equality in the following theorem was conjectured by Chapoton. In fact, we obtain a simple closed formula during the proof.

Theorem 2.2. *For* $u \geq 2$, *the generating series for the Zeta polynomials of* P_{t_n} *is*

$$1 + \sum_{n\geq 1} Z_{t_n}(u, 1)s^n = \exp\left(\int_0^s \frac{u}{(1-ux)(1+x-ux)}\,dx\right) = \left(\frac{1-su+s}{1-su}\right)^u.$$

Proof. We work in the ring of formal power series. Manipulations (taking logarithms, differentiating, and exponentiating) are performed in this ring.

For the arbor t on one element, it is clear that the Zeta polynomial is $1 + (u-1)X$. Therefore, for $n-1$ leaves, we obtain

$$Z_W = (1 + (u-1)X)^{n-1} = \sum_{k=0}^{n-1} W_k X^k, \quad \text{where} \quad W_k = \binom{n-1}{k}(u-1)^k.$$

By Proposition 2.1, for $0 \leq j \leq n$, the coefficient of X^j in $Z_{t_n}(u, X)$ is

$$\sum_{\ell=\max(0,j-n+1)}^{j} \binom{u-2+\ell}{\ell} W_{j-\ell} = \sum_{\ell=0}^{\min(j,n-1)} \binom{u-2+j-\ell}{j-\ell} W_\ell = \sum_{\ell=0}^{j} \binom{u-2+j-\ell}{j-\ell} W_\ell,$$

with the convention that $W_\ell = 0$ if $\ell \notin \{0, 1, \ldots, n-1\}$. Summing over all possible heights $0 \leq j \leq n$, we get

$$Z_{t_n}(u, 1) = \sum_{j=0}^{n} \sum_{\ell=0}^{j} \binom{u-2+j-\ell}{j-\ell} W_\ell.$$

We extract the coefficients of W_k $(0 \le k \le n-1)$ and obtain

$$\sum_{j=k}^{n} \binom{u-2+j-k}{j-k} = \sum_{\ell=0}^{n-k} \binom{u-2+\ell}{\ell} = \binom{u+n-k-1}{n-k}.$$

Therefore, we have

$$\begin{aligned} Z_{t_n}(u,1) &= \sum_{k=0}^{n-1} W_k \binom{u+n-k-1}{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (u-1)^k \binom{u+n-k-1}{n-k} \\ &= \sum_{k=0}^{\infty} \binom{n-1}{k} (u-1)^k \binom{u+n-k-1}{n-k}. \end{aligned}$$

Let $G(s) = 1 + \sum_{n\ge1} Z_{t_n}(u,1)s^n$. By making the substitution $m = n-k-1$, and given that $n \ge 1$ and $k \ge 0$ and $\binom{n-1}{k} = 0$ if $n-k-1 < 0$, the summation range for m can be restricted to $[0,\infty)$ without altering the final sum. We get

$$\begin{aligned} G(s) &= 1 + \sum_{n\ge1}\sum_{k\ge0} \binom{n-1}{k} (u-1)^k \binom{u+n-k-1}{n-k} s^n \\ &= 1 + \sum_{m\ge0}\sum_{k\ge0} \binom{k+m}{k} (u-1)^k \binom{u+m}{m+1} s^{k+m+1} \\ &= 1 + \sum_{m\ge0} s^{m+1} \binom{u+m}{m+1} \left[\sum_{k\ge0} \binom{k+m}{k} (s(u-1))^k \right] \\ &= 1 + \sum_{m\ge0} \binom{u+m}{m+1} \frac{s^{m+1}}{(1-su+s)^{m+1}} = \sum_{m\ge0} \binom{u+m-1}{m} \frac{s^m}{(1-su+s)^m} \\ &= \left(\frac{1}{1-\frac{s}{1-su+s}} \right)^u = \left(\frac{1-su+s}{1-su} \right)^u. \end{aligned}$$

Taking the logarithm and differentiating with respect to s, we obtain

$$\frac{d}{ds} \ln G(s) = u \frac{d}{ds} \Big(\ln(1-us+s) - \ln(1-us) \Big) = \frac{u}{(1-us)(1+s-us)}.$$

Integrating both sides and exponentiating yields the desired result. □

2.2 M-triangles

Given a poset P, the *Möbius function* of P (denoted μ_P) is defined by

$$\mu_P(a,a) = 1, \quad \text{for all} \quad a \in P;$$

$$\mu_P(a,b) = -\sum_{a \le e < b} \mu_P(a,e), \quad \text{for all} \quad a < b \quad \text{in} \quad P.$$

Given a graded poset P, one can define the *M-triangle* of P as the following sum of Möbius numbers weighted using the rank function ht

$$M_P(X,Y) = \sum_{a \le b} \mu_P(a,b) X^{\mathrm{ht}(a)} Y^{\mathrm{ht}(b)}.$$

For an arbor t, in order to obtain $M_{P_t}(X,Y)$, Chapoton introduced the polynomial

$$K_t(X,Y) = \sum_{a \in P_t} X^{\mathrm{nz}(a)} Y^{\mathrm{ht}(a)},$$

where $\mathrm{nz}(a)$ is the number of non-zero coordinates in a. For the arbor t on one element, it is clear that $K_t(X,Y) = 1 + XY$. Let t be an arbor on $[n]$ with $n \ge 2$. Let $(t^{(1)}, \ldots, t^{(w)})$ be the sub-trees of the root vertex of t. Let r be the size of the root vertex of t. Chapoton obtained the following characterization of the coefficients of the polynomial $K_t(X,Y)$.

Proposition 2.3. [4, Proposition 2.8] *Let $W(X,Y) = \prod_{k=1}^{w} K_{t^{(k)}}(X,Y)$ and write $W(X,Y) = \sum_{i,j} W_{i,j} X^i Y^j$. Then for $0 \le j \le k \le n$, the coefficient of $X^j Y^k$ in $K_t(X,Y)$ is*

$$\sum_{\ell=\max(0,j-n+r)}^{\min(j,r)} \sum_{m=\max(\ell,k-n+r)}^{k+\ell-j} \binom{r}{\ell} \binom{m-1}{m-\ell} W_{j-\ell,k-m}.$$

Furthermore, Chapoton derived a transformation formula from $K_t(X,Y)$ to $M_t(X,Y)$.

Proposition 2.4. [4, Proposition 2.10] *For every arbor t, we have*

$$M_t(X,Y) = K_t(1 - 1/X, XY).$$

In the particular case t_n under consideration, we obtain a reduced formula.

Proposition 2.5. *For any $n \in \mathbb{Z}_+$, the polynomial $K_{t_n}(X,Y)$ is given by*

$$K_{t_n}(X,Y) = (1+XY)^n + \frac{XY^2}{1-Y}(1+XY)^{n-1} - \frac{XY^2}{1-Y}(Y(1+X))^{n-1}.$$

Proof. For $0 \le j \le k \le n$, we consider the coefficient of $X^j Y^k$ in $K_{t_n}(X,Y)$.

If $j = k$, then the enumerated $a \in P_{t_n}$ possesses property $\mathrm{nz}(a) = \mathrm{ht}(a)$, so a has exactly j non-zero components, all of which are equal to 1. One can easily see that there exist exactly $\binom{n}{j}$ such a.

If $j < k$, it follows that the enumerated a takes the value $k-j+1$ at the component of the root (which means $j \ge 1$), and among the remaining $n-1$ components, it possesses exactly $j-1$ non-zero entries, all of which are equal to 1. There exist $\binom{n-1}{j-1}$ such a.

This yields the explicit expression for $K_{t_n}(X,Y)$ as follows:

$$\begin{aligned}
K_{t_n}(X,Y) &= \sum_{j=0}^{n}\binom{n}{j}X^jY^j + \sum_{1\le j<k\le n}\binom{n-1}{j-1}X^jY^k \\
&= \sum_{j=0}^{n}\binom{n}{j}(XY)^j + \sum_{j=1}^{n-1}\binom{n-1}{j-1}X^j\sum_{k=j+1}^{n}Y^k \\
&= (1+XY)^n + \frac{XY^2}{1-Y}\sum_{j=1}^{n-1}\binom{n-1}{j-1}(XY)^{j-1} - \frac{XY^{n+1}}{1-Y}\sum_{j=1}^{n-1}\binom{n-1}{j-1}X^{j-1} \\
&= (1+XY)^n + \frac{XY^2}{1-Y}(1+XY)^{n-1} - \frac{XY^2}{1-Y}(Y(1+X))^{n-1}.
\end{aligned}$$

This completes the proof. □

Theorem 2.6. (Conjectured by Chapoton) *The generating series for the M-triangles of P_{t_n} is*

$$1+\sum_{n\ge1}M_{t_n}(X,Y)s^n = \frac{(XYs-Ys-1)(XYs-1)}{(2XYs-Ys-1)(XYs-Ys+s-1)}.$$

Proof. By Proposition 2.5 and Proposition 2.4, we have

$$\begin{aligned}
&M_{t_n}(X,Y) = K_{t_n}(1-1/X, XY) \\
=&(1+XY-Y)^n + \frac{X^2Y^2-XY^2}{1-XY}(1+XY-Y)^{n-1} - \frac{X^2Y^2-XY^2}{1-XY}(2XY-Y)^{n-1}.
\end{aligned}$$

Defining the substituted variables

$$A = 1+XY-Y, \quad B = 2XY-Y, \quad \text{and} \quad C = \frac{X^2Y^2-XY^2}{1-XY},$$

we get

$$M_{t_n}(X,Y) = A^n + CA^{n-1} - CB^{n-1}.$$

Now, by computing the generating function $G(s) = 1+\sum_{n\ge1}M_{t_n}(X,Y)s^n$, we obtain

$$\begin{aligned}
G(s) &= 1+\sum_{n\ge1}(As)^n + Cs\sum_{n\ge1}A^{n-1}s^{n-1} - Cs\sum_{n\ge1}B^{n-1}s^{n-1} \\
&= 1+\frac{As}{1-As}+\frac{Cs}{1-As}-\frac{Cs}{1-Bs} \\
&= \frac{(XYs-Ys-1)(XYs-1)}{(2XYs-Ys-1)(XYs-Ys+s-1)}.
\end{aligned}$$

This completes the proof. □

3 Generating Functions for Polytope Invariants

3.1 Ehrhart Polynomials

Let $\mathcal{P}$ be a d-dimensional lattice convex polytope in $\mathbb{R}^n$. The *Ehrhart function*

$$E_{\mathcal{P}}(u) = |u\mathcal{P} \cap \mathbb{Z}^n|, \quad u = 1, 2, \ldots,$$

counts the number of integer points in the u-th dilation $u\mathcal{P} = \{u\alpha : \alpha \in \mathcal{P}\}$ of $\mathcal{P}$.

Ehrhart's famous theorem [6] says that $E_{\mathcal{P}}(u)$ is a polynomial in u of degree d, now known as the *Ehrhart polynomial* of $\mathcal{P}$. It is well-known [3, Corollary 3.20; Theorem 5.6] that the leading coefficient of $E_{\mathcal{P}}(u)$ equals the (relative) volume of $\mathcal{P}$, the second highest coefficient of $E_{\mathcal{P}}(u)$ equals half of the boundary volume of $\mathcal{P}$, and the constant term is 1.

Theorem 3.1. (Conjectured by Chapoton) *The generating series for the Ehrhart polynomials of Q_{t_n} is*

$$1 + \sum_{n \geq 1} E_{t_n}(u) s^n = \frac{1}{2}\left(1 - \frac{s}{(us + s - 1)} - \frac{s - 1}{(us + s - 1)^2}\right).$$

First proof of Theorem 3.1. For every integer $m \geq 1$, Chapoton [4] introduced the height distribution polynomial $F_{t,m}(X)$ that counts lattice points in the dilated polytope mQ_t with weight X^h at height h, which is multiplicative with respect to the direct product of polytopes.

Fix an integer $m \geq 1$ and consider the dilated polytope mQ_{t_n}.

The arbor t_n consists of a root of size 1 and $n-1$ leaves of size 1. For a single leaf, the height distribution polynomial is $F_{t_1,m}(X) = 1 + X + \cdots + X^m$. For $n-1$ independent leaves, their joint distribution is $F_{\prod_{i=1}^{n-1} t_1,m}(X) = (1 + X + \cdots + X^m)^{n-1} = \sum_{k=0}^{m(n-1)} w_k X^k$.

Obviously, the number of lattice points at height j in mQ_{t_n} is $\sum_{k=0}^{\min(j,m(n-1))} w_k$. Summing over all possible heights $0 \leq j \leq mn$, the Ehrhart polynomial $E_{t_n}(u)$ evaluated at m gives

$$\begin{aligned}
E_{t_n}(m) &= \sum_{j=0}^{mn} \left(\sum_{k=0}^{\min(j,m(n-1))} w_k \right) = \sum_{k=0}^{m(n-1)} (mn - k + 1) w_k \\
&= (mn+1) \sum_{k=0}^{m(n-1)} w_k - \sum_{k=0}^{m(n-1)} k w_k \\
&= (mn+1) F_{\prod_{i=1}^{n-1} t_1,m}(1) - F'_{\prod_{i=1}^{n-1} t_1,m}(1) \\
&= (m+1)^{n-1} \left(\frac{mn}{2} + \frac{m}{2} + 1 \right).
\end{aligned}$$

Replacing m with the continuous variable u, we construct the generating function $G(s, u) =$

$1 + \sum_{n\geq 1} E_{t_n}(u)s^n$. Then we have

$$\begin{aligned} G(s,u) &= 1 + \sum_{n\geq 1}(u+1)^{n-1}\left(\frac{un}{2} + \frac{u}{2} + 1\right)s^n \\ &= 1 + \frac{\frac{u}{2}+1}{u+1}\sum_{n\geq 1}\left((u+1)s\right)^n + \frac{u}{2(u+1)}\sum_{n\geq 1} n((u+1)s)^n \\ &= \frac{1}{2}\left(1 - \frac{s}{us+s-1} - \frac{s-1}{(us+s-1)^2}\right). \end{aligned}$$

This completes the proof. □

Second proof of Theorem 3.1. In [2], the Ehrhart polynomial of the arbor polytope t_n is given by

$$E_{t_n}(u) = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\binom{(n-j)(u+1)}{n}.$$

For the sake of completeness, we briefly outline the proof of this formula due to Athanasiadis et al. [2]. Observe that $E_{t_n}(u)$ counts the lattice points $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ satisfying

$$x_1 + x_2 + \cdots + x_n \leq un, \quad \text{and} \quad 0 \leq x_i \leq u \quad \text{for} \quad 1 \leq i \leq n-1.$$

Since the number of points $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ which satisfy the first inequality and violate the second for any j given values of $i \in [n-1]$ is equal to $\binom{(n-j)(u+1)}{n}$. By the principle of inclusion–exclusion, we therefore obtain

$$E_{t_n}(u) = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\binom{(n-j)(u+1)}{n},$$

as desired.

Let $[x^m]f(x)$ be defined as the coefficient of x^m in the polynomial $f(x)$. Set $a = u+1$, we obtain

$$\begin{aligned} E_{t_n}(u) &= \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\binom{(n-j)a}{n} = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}[x^n](1+x)^{a(n-j)} \\ &= [x^n](1+x)^{an}\sum_{j=0}^{n-1}\binom{n-1}{j}(-(1+x)^{-a})^j \\ &= [x^n](1+x)^{an}(1-(1+x)^{-a})^{n-1} = [x^n](1+x)^a((1+x)^a-1)^{n-1} \\ &= [x^n](1+x)^a x^{n-1}(g(x))^{n-1} = [x](1+x)^a(g(x))^{n-1}, \end{aligned}$$

where $g(x) = \frac{(1+x)^a-1}{x}$. Therefore, we have

$$1 + \sum_{n=1}^{\infty} E_{t_n}(u)s^n = 1 + s\sum_{n=1}^{\infty}[x](1+x)^a(s\cdot g(x))^{n-1} = 1 + s[x]\frac{(1+x)^a}{1-s\cdot g(x)}.$$

Let $G(x) = \frac{(1+x)^a}{1-s\cdot g(x)}$ and $h(x) = 1 - s \cdot g(x)$. We have $[x]G(x) = G'(0)$. According to

$$g(0) = a, \quad g'(0) = \frac{a(a-1)}{2}, \quad h(0) = 1 - as, \quad h'(0) = -sg'(0) = \frac{-sa(a-1)}{2},$$

we get

$$G'(0) = \frac{a \cdot h(0) - h'(0)}{(h(0))^2} = \frac{a - \frac{a(a+1)s}{2}}{(1-as)^2}.$$

Hence, we conclude that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} E_{t_n}(u)s^n = 1 + s \cdot G'(0) &= \frac{1 - as + \frac{a(a-1)s^2}{2}}{(1-as)^2} \\ &= \frac{1}{2}\left(1 - \frac{s}{as-1} - \frac{s-1}{(as-1)^2}\right) \\ &= \frac{1}{2}\left(1 - \frac{s}{(us+s-1)} - \frac{s-1}{(us+s-1)^2}\right). \end{aligned}$$

This completes the proof. □

3.2 Laplace Transform of Volume Functions

For a given lattice polytope $\mathcal{P}$, we know that the leading coefficient of $E_{\mathcal{P}}(u)$ equals the volume of $\mathcal{P}$. For an arbor t, Chapoton refines the volume of Q_t into a volume function $V_t(h)$, that records the volume of every slice of Q_t according to the height function ht. This function vanishes outside the segment $[0, n]$, where n is the size of t. For example, for the arbor t on one element, $V_t(h) = 1$ for $0 \leq h \leq 1$ and 0 otherwise.

To be precise regarding the normalization of the slice volume, $V_t(h)$ is defined as the density of the push-forward of Lebesgue measure restricted to Q_t under the height map $\mathrm{ht}(x) = \sum_{i\in[n]} x_i$. This definition naturally ensures that $\int_0^n V_t(h)\, dh$ recovers the exact n-dimensional volume of Q_t.

The *Laplace transform* of the volume function $V_t(h)$ is defined by

$$\mathcal{L}_t(v) = \int_0^{\infty} \exp(-vh)V_t(h)dh.$$

Note that the value at $v = 0$ recovers the volume of Q_t.

Let $\mathbf{T}_n$ denote the operator that truncates a function of the variable h to the segment $[0, n]$, extended by zero outside of this segment. By setting $E = e^{-v}$ and $V = 1/v$, this truncation operation naturally induces a linear action on polynomials in E and V, which we still denote by the same notation $\mathbf{T}_n$. Let $\mathcal{L}^{-1}$ denote the inverse Laplace transform. The induced action on a monomial E^kV^l is defined by the following integral:

$$\mathbf{T}_n(E^kV^l) := \left(\int_0^n \exp(-vh)\mathcal{L}^{-1}\left(e^{-kv}v^{-l}\right)(h)\, dh\right)\Big|_{e^{-v}\to E,\ 1/v\to V}.$$

This definition describes a sequence of transformations: one substitutes the variables to form a function of the variable v, applies the inverse Laplace transform to recover the corresponding function of h, computes its Laplace transform bounded over the interval $[0, n]$, and finally expresses the resulting function of v algebraically back in terms of E and V.

Chapoton [4, Section 2.3] introduced a bivariate polynomial $L_t(E, V)$ for computing the Laplace transform $\mathcal{L}_t(v)$ where $E = e^{-v}$ and $V = 1/v$. For a given arbor t, one can compute $\mathcal{L}_t(v)$ by first computing polynomial $L_t(E, V)$, and then replace E by e^{-v} and V by $1/v$. To recover the volume, one needs to take the limit at $v = 0$ of $L_t(e^{-v}, 1/v)$. Chapoton also derived a formula which translates the action of $\mathbf{T}_n$ on polynomials in E and V.

Proposition 3.2. [4, Lemma 2.5] *For integers $k \geq 0$ and $\ell \geq 0$, if $\ell \geq n$, then $\mathbf{T}_n(V^{k+1}E^\ell) = 0$. Otherwise,*

$$\mathbf{T}_n(V^{k+1}E^\ell) = V^{k+1}E^\ell - \sum_{j=0}^{k} \frac{(n-\ell)^{k-j}}{(k-j)!} V^{j+1}E^n.$$

The initial value $L_t(E, V)$ for the arbor of size 1 is $-VE + V$. Let t be an arbor on the set $[n]$ with $n \geq 2$. Let $(t^{(1)}, \ldots, t^{(w)})$ be the sub-trees of the root vertex of t. Let r be the size of the root vertex of t. The polynomial $L_t(E, V)$ is

$$\mathbf{T}_n\left(\prod_{k=1}^{w} L_{t^{(k)}}(E, V) \cdot \mathbf{T}_n(V^r)\right). \tag{1}$$

Theorem 3.3. (Conjectured by Chapoton) *The generating series for the Laplace transforms of volume functions of Q_{t_n} is*

$$\sum_{n\geq 1} L_{t_n}(E, V)s^n = V\left(\frac{s}{EVs - Vs + 1} + \frac{Es}{Es - 1}\right).$$

Proof. By Equation (1), the Laplace transform of the volume function follows the recursion

$$L_{t_n}(E, V) = \mathbf{T}_n\left(L_{t_1}^{n-1} \cdot \mathbf{T}_n(V)\right),$$

where $L_{t_1}(E, V) = V(1 - E)$. According to Proposition 3.2, $\mathbf{T}_n(V) = V - VE^n$. Substituting this into the recursion gives

$$L_{t_n}(E, V) = \mathbf{T}_n\left(V^n(1 - E)^{n-1} - V^n(1 - E)^{n-1}E^n\right).$$

The truncation operator $\mathbf{T}_n$ maps $V^kE^\ell \mapsto 0$ for $\ell \geq n$. Since the expanded terms of $V^n(1 - E)^{n-1}E^n$ all possess an E degree of at least n, $\mathbf{T}_n$ annihilates this entire section. Thus, we only need to evaluate $\mathbf{T}_n\left(V^n(1 - E)^{n-1}\right)$. Using the binomial expansion and Proposition 3.2, we obtain

$$L_{t_n}(E, V) = \mathbf{T}_n\left(\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^\ell V^n E^\ell\right)$$

$$= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^{\ell} \left(V^n E^{\ell} - \sum_{j=0}^{n-1} \frac{(n-\ell)^{n-1-j}}{(n-1-j)!} V^{j+1} E^n \right)$$
$$= V^n (1-E)^{n-1} - E^n \sum_{j=0}^{n-1} V^{j+1} \left[\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^{\ell} \frac{(n-\ell)^{n-1-j}}{(n-1-j)!} \right].$$

Let Δ denote the standard forward difference operator, defined for any function $P(x)$ by $\Delta P(x) = P(x+1) - P(x)$. The inner bracketed sum can be evaluated as $\Delta^{n-1} P(1)$ for the polynomial $P(x) = \frac{x^{n-1-j}}{(n-1-j)!}$. Since $P(x)$ is of degree $n-1-j$, its $(n-1)$-th forward difference evaluates to the Kronecker delta $\delta_{j,0}$ (where $\delta_{j,0} = 1$ if $j = 0$, and 0 if $j > 0$). Consequently, the double sum collapses to a single term

$$L_{t_n}(E, V) = V^n (1-E)^{n-1} - V E^n.$$

Generating the formal power series over n:

$$\begin{aligned} G(s) &= \sum_{n \geq 1} \left(V^n (1-E)^{n-1} - V E^n \right) s^n \\ &= Vs \sum_{n \geq 1} (Vs(1-E))^{n-1} - V \sum_{n \geq 1} (Es)^n \\ &= V \left(\frac{s}{EVs - Vs + 1} + \frac{Es}{Es - 1} \right). \end{aligned}$$

This completes the proof. □

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